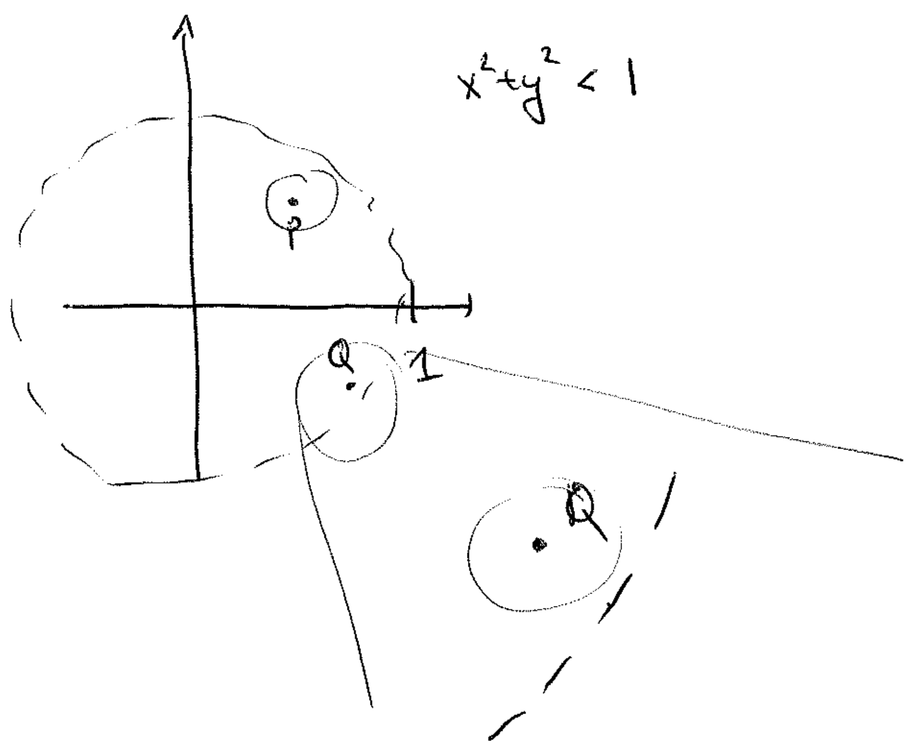
A vector field $F(\vec{x}, y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$
is <u>conservative</u> (or is a <u>gradient</u> <u>vector field</u>)
if $\vec{F}(x,y) = \nabla f(x,y)$ for some scalar function
f (the <u>potential</u>)

open regions



<u>not open</u>

Puzzler:

Suppose $f(x,y)$ is a ~~scal~~ scalar function
and $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$

what is $\frac{d}{dt} f(x(t), y(t))$?

$$\begin{tikzcd} & f \arrow[dl] \arrow[dr] & \\ x \arrow[d] & & y \arrow[d] \\ t & & t \end{tikzcd}$$

① $\frac{d}{dt} f(x(t), y(t)) =$

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= \vec{\nabla} f \cdot \vec{r}\,'(t)$$

$$\frac{d}{dt} f(x(t), y(t)) = \nabla f(x(t), y(t)) \cdot \langle x'(t), y'(t) \rangle$$

② $$\int_a^b F'(t)\,dt = F(b) - F(a)$$

Suppose $\vec{F}(x,y) = \vec{\nabla} f(x,y) \qquad \vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$

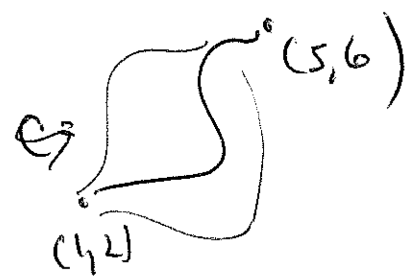
$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \nabla f(x(t), y(t)) \cdot \langle x'(t), y'(t) \rangle \, dt$$

$$= \int_a^b \frac{d}{dt}\left[f(x(t), y(t)) \right] dt$$

$$= f(x(b), y(b)) - f(x(a), y(a))$$

$f(x,y) = x^2 + y^2$

$\nabla f = 2x\,\hat{\imath} + 2y\,\hat{\jmath}$



$\int_C \nabla f \cdot d\vec{r} =$

$f(5,6) - f(1,2) =$

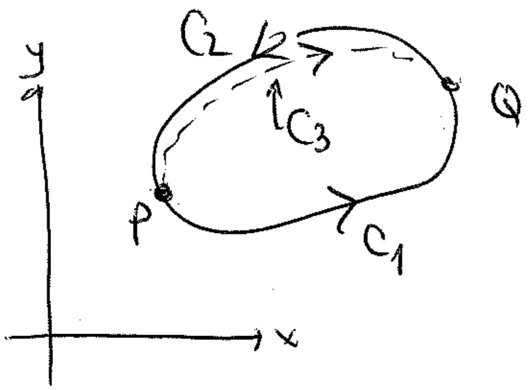
$(5^2+6^2) - (1^2+2^2) =$

(1) $\int_C \vec{F}\cdot\vec{dr}$ is independent of path

(2) $\int_C \vec{F}\cdot\vec{dr} = 0$ for any closed path

---

(1) $\Rightarrow$ (2) : Take a closed path

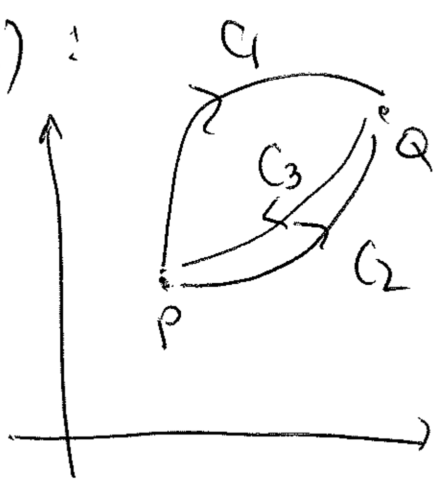


$$\int_{C_1} \vec{F}\cdot\vec{dr} + \int_{C_2} \vec{F}\cdot\vec{dr}$$

$$= \int_{C_1} \vec{F}\cdot\vec{dr} - \int_{C_3} \vec{F}\cdot dr$$

$$= 0$$

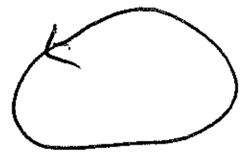
(2) $\Rightarrow$ (1) :



$$\int_{C_1} \vec{F}\cdot\vec{dr} + \int_{C_3} \vec{F}\cdot\vec{dr} = 0$$

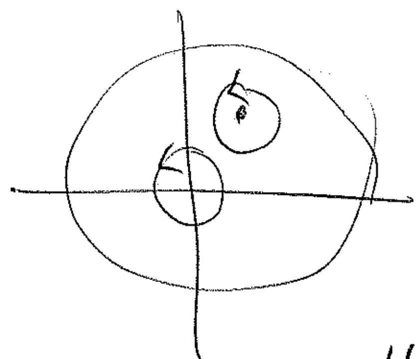
$$\int_{C_1} \vec{F}\cdot\vec{dr} - \int_{C_2} \vec{F}\cdot\vec{dr} = 0$$

Simple curve

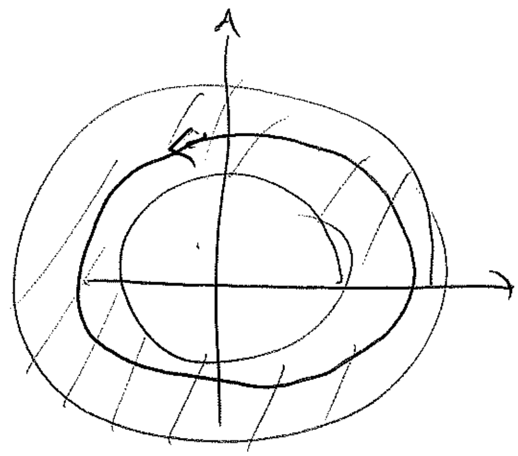


Non-simple-curve

Simply connected:
"no holes"



Not simply connected → "has holes"



$$\vec{F}(x,y) = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$$
$$= \frac{\partial f}{\partial x}(x,y)\,\hat{i} + \frac{\partial f}{\partial y}(x,y)\,\hat{j}$$

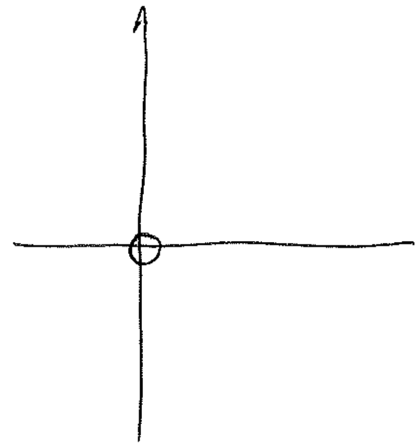
$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}(x,y)$$
$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}(x,y)$$

	P	Q	$\frac{\partial P}{\partial y}$	$\frac{\partial Q}{\partial x}$
$\vec{F}(x,y) = -y\,\hat{i} + x\,\hat{j}$	$-y$	x	-1	$+1$
$\vec{F}(x,y) = x^3\,\hat{i} + y^3\,\hat{j}$	x^3	y^3	0	0
$\vec{F}(x,y) = ye^x\,\hat{i} + (e^x + e^y)\,\hat{j}$	ye^x	$e^x + e^y$	e^x	e^x
$\vec{F}(x,y) = \frac{-y}{x^2+y^2}\,\hat{i} + \frac{x}{x^2+y^2}\,\hat{j}$			$\frac{y^2-x^2}{(x^2+y^2)^2}$	$\frac{y^2-x^2}{(x^2+y^4)^2}$

$$\frac{-1}{x^2+y^2} + - \frac{-y \cdot 2y}{(x^2+y^2)^2}$$

$\{(x,y): x^2+y^2 \neq 0\}$

Not simply connected!



$$\int_C \vec{F}\cdot d\vec{r} \qquad C: x^2+y^2=1$$

$$x(t) = \cos t \qquad 0 \le t \le 2\pi$$
$$y(t) = \sin t$$

$$\vec{r}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$$

$$\vec{F}(\vec{r}(t)) = -y(t)\,\hat{i} + x(t)\,\hat{j}$$
$$= -\sin(t)\,\hat{i} + \cos(t)\,\hat{j}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} [-\sin(t)\,\hat{i} + \cos(t)\,\hat{j}]\cdot[-\sin t\,\hat{i} + \cos t\,\hat{j}]\,dt$$
$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$
$$= \int_0^{2\pi} dt$$
$$= 2\pi$$

$$\vec{F}(x,y) = \underset{+}{\underset{\shortparallel}{\overset{P}{(y^2-2x)}}}\,\hat{\imath} + \underset{\#}{\underset{\shortparallel}{\overset{Q}{2xy}}}\,\hat{\jmath}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \; ?$$

$$2y = 2y \; ✓$$

$$\nabla f(x,y) = \underset{+}{\frac{\partial f}{\partial x}}\,\hat{\imath} + \underset{\#}{\frac{\partial f}{\partial y}}\,\hat{\jmath}$$

$+$ 1
$$\frac{\partial f}{\partial x} = y^2 - 2x$$

$$f(x,y) = y^2 \cdot x - x^2 + C(y)$$

$\#$ 2
$$\frac{\partial f}{\partial y} = 2xy$$

$$\cancel{2xy} - 0 + C'(y) = \cancel{2xy}$$

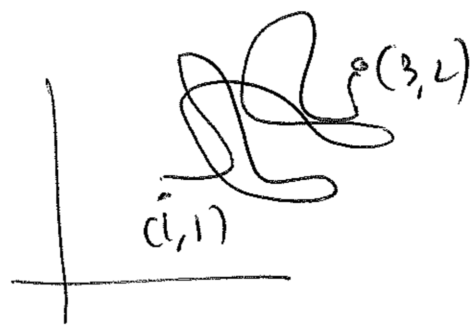
$$C'(y) = 0$$

$$C(y) = C \quad (\text{constant})$$

So $f(x,y) = y^2x - x^2 + C$

$$\vec{F}(x,y) = (y^2-2x)\hat{i} + 2xy\,\hat{j} = \nabla f(x,y)$$

$$\int_C \vec{F}\cdot d\vec{r} =$$



$$f(3,2) - f(1,1) =$$

$$\left(2^2\cdot 3 - 3^2 + \cancel{C}\right) - \left(1 - 1 + \cancel{C}\right) =$$

---

find $\displaystyle\int_{(1,1)}^{(2,2)} \underbrace{\left(x^2\hat{i} + y^2\hat{j}\right)}_{=\vec{F}(x,y)}\cdot d\vec{r}$

$$\vec{F}(x,y) = \underbrace{x^2}_{=P}\hat{i} + \underbrace{y^2}_{=Q}\hat{j}$$

$$\frac{\partial P}{\partial y} = 0 \qquad \frac{\partial Q}{\partial x} = 0$$

$$x^2\hat{i} + y^2\hat{j} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

1. $$\frac{\partial f}{\partial x}(x,y) = x^2$$

$$f(x,y) = \frac{x^3}{3} + C(y)$$

2. $$\frac{\partial f}{\partial y} = y^2$$

$$0 + C'(y) = y^2$$

$$C(y) = \frac{y^3}{3}$$

$$f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3$$

C is the path from (0, 0) to (2, 3)

$$\int_C \vec{F}\cdot d\vec{r} = f(2,3) - f(0,0) = \frac{1}{3}(2^3+3^3)$$