Calculus I Meets Calculus III

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University of Kentucky

December 3, 2018

Homework

- Begin Homework D3 due Wednesday, December 5
- Review for Final Exam on Thursday, December 13, 6:00-8:00 PM
- Be sure you know which room to go to for the final!

Final Exam Room Assignments

Sections 001-008	BE 111
Sections 009-013	**BS 107**
Sections 014-016	KAS 213

The Big Picture

Unit I	Vectors and Space Curves
Unit II	Differential Calculus
Unit III	Double and Triple Integrals
Unit IV	Calculus of Vector Fields

Goals of the Day

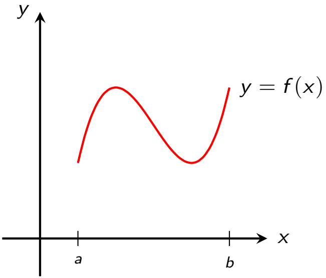
- Review what you learned in Calculus I
- Connect it with what you learned in Calculus III
- Organize your review for the Final

Today we'll mainly talk about *differential* calculus

Calculus I

Calculus I was about *functions of one variable*

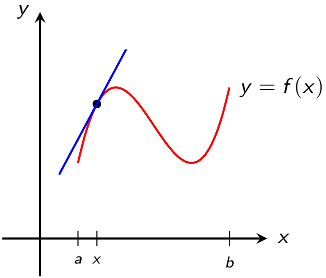
Calculus I



Calculus I was about *functions of one variable*

- The *graph* of a function $y = f(x)$ is a curve in the xy plane with points $(x, f(x))$

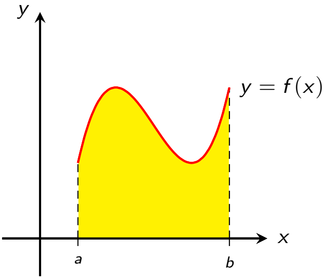
Calculus I



Calculus I was about *functions of one variable*

- The *graph* of a function $y = f(x)$ is a curve in the xy plane with points $(x, f(x))$
- The *derivative* $f'(x)$ of a function $f(x)$ is the slope of the tangent line to the graph of f at $(x, f(x))$

Calculus I



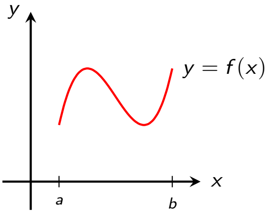
Calculus I was about *functions of one variable*

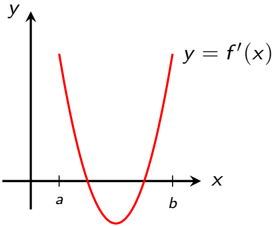
- The *graph* of a function $y = f(x)$ is a curve in the xy plane with points $(x, f(x))$
- The *derivative* $f'(x)$ of a function $f(x)$ is the slope of the tangent line to the graph of f at $(x, f(x))$
- The *integral*

$$\int_a^b f(x)\,dx$$

gives the (net) area under the graph of f between $x = a$ and $x = b$

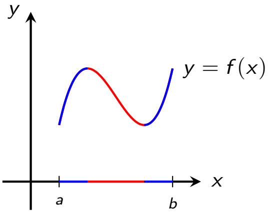
The Derivative - Calculus I

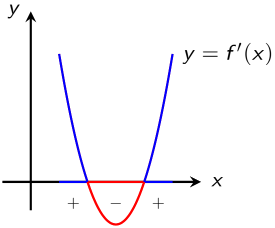




The derivative $f'(x)$ of a function $f(x)$ tells us:

The Derivative - Calculus I

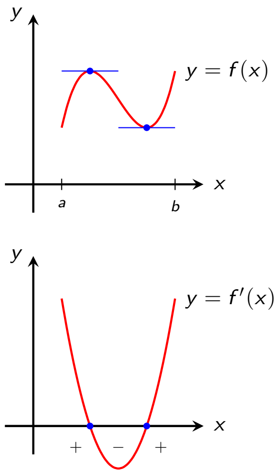




The derivative $f'(x)$ of a function $f(x)$ tells us:

- When a function is *increasing* or *decreasing*

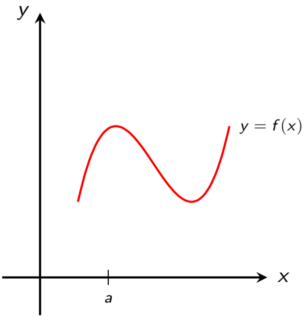
The Derivative - Calculus I



The derivative $f'(x)$ of a function $f(x)$ tells us:

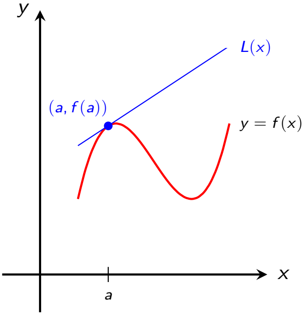
- When a function is *increasing* or *decreasing*
- When a function has *local maxima* or *local minima*

The Derivative - Calculus I



The derivative $f'(a)$ of a function $f(x)$ also gives us a *linear approximation* to the function f near $x = a$:

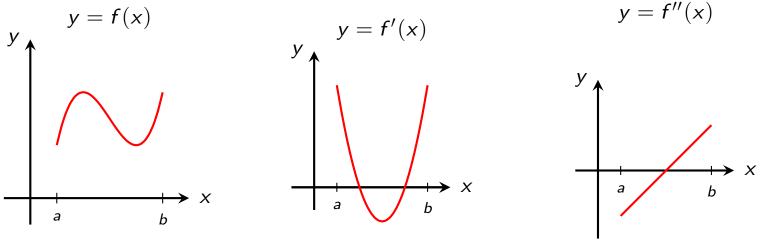
The Derivative - Calculus I



The derivative $f'(a)$ of a function $f(x)$ also gives us a *linear approximation* to the function f near $x = a$:

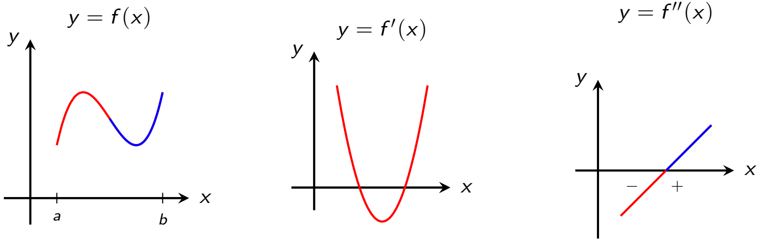
$$L(x) = f(a) + f'(a)(x - a)$$

The Second Derivative - Calculus I



The second derivative $f''(x)$ tells us:

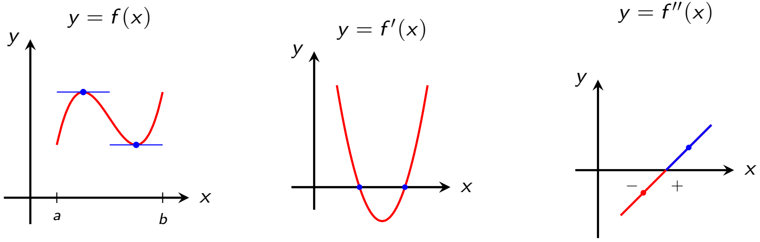
The Second Derivative - Calculus I



The second derivative $f''(x)$ tells us:

- When the graph of f is concave up, and when it is concave down

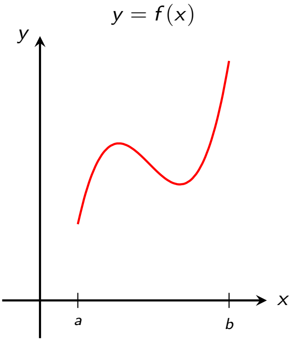
The Second Derivative - Calculus I



The second derivative $f''(x)$ tells us:

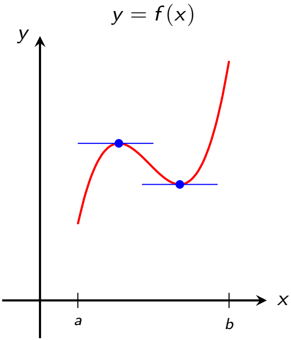
- When the graph of f is concave up, and when it is concave down
- When critical points are local maxima, and when they are local minima

Optimization - Calculus I



To find the absolute maximum and absolute minimum of a function $f(x)$ on an interval $[a, b]$:

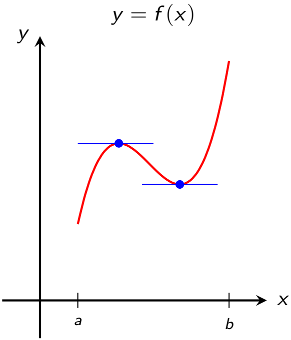
Optimization - Calculus I



To find the absolute maximum and absolute minimum of a function $f(x)$ on an interval $[a, b]$:

1. Find the *interior critical points* of f

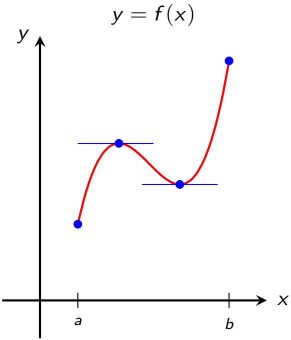
Optimization - Calculus I



To find the absolute maximum and absolute minimum of a function $f(x)$ on an interval $[a, b]$:

1. Find the *interior critical points* of f
2. Test f at the interior critical points

Optimization - Calculus I



To find the absolute maximum and absolute minimum of a function $f(x)$ on an interval $[a, b]$:

1. Find the *interior critical points* of f
2. Test f at the interior critical points
3. Test f at the endpoints a and b

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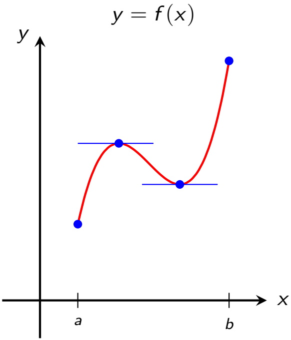


To find the absolute maximum and absolute minimum of a function $f(x)$ on an interval $[a, b]$:

1. Find the *interior critical points* of f
2. Test f at the interior critical points
3. Test f at the endpoints a and b
4. The largest value of f in the list is its absolute maximum, and the smallest value of f in the list is its absolute minimum

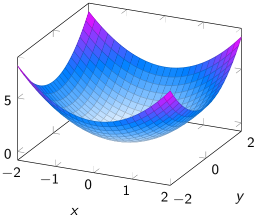
Why Calculus I is So Easy

- There is only one variable to change, and so only one *rate* of change
- The domain of a function of one variable is typically an interval with two endpoints

Calculus III

Calculus III is about functions of *two* (or more) variables

$f(x) = x^2 + y^2$



Calculus III

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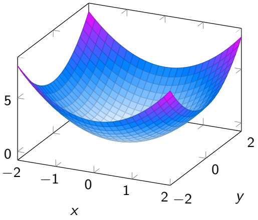


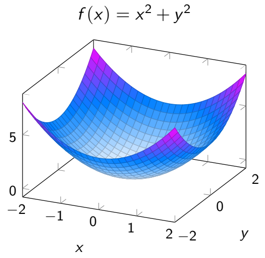
Calculus III is about functions of *two* (or more) variables

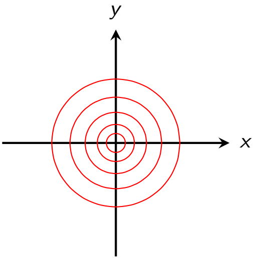
- The *graph* of a function

$$z = f(x, y)$$

is a *surface* in *xyz* space with points $(x, y, f(x, y))$

Calculus III





Calculus III is about functions of *two* (or more) variables

- The *graph* of a function

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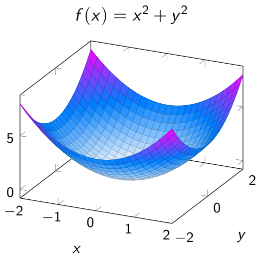
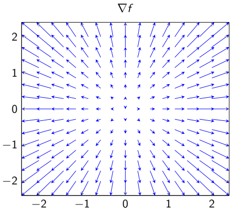
- You can also visualize a function of two variables through its *contour plot*

Calculus III





Calculus III is about functions of *two* (or more) variables

- The *graph* of a function

$$z = f(x, y)$$

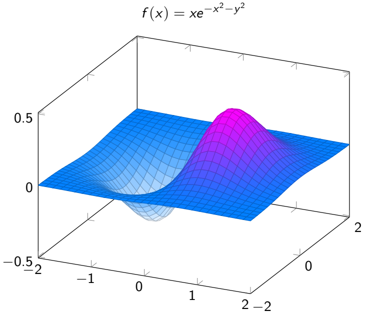
  is a *surface* in *xyz* space with points $(x, y, f(x, y))$
- You can also visualize a function of two variables through its *contour plot*
- The *derivative* of a function of two variables is the *gradient vector*

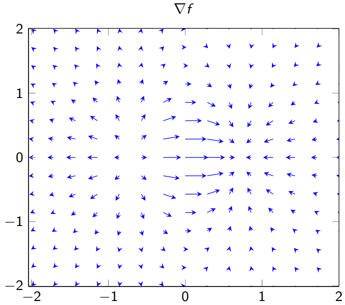
$$(\nabla f)(x, y) = \left\langle \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right\rangle$$

The Derivative - Calculus III

The gradient vector $(\nabla f)(a, b)$:

- Has magnitude equal to the maximum rate of change of f at (a, b)
- Points in the direction of greatest change of f at (a, b)
- Is the zero vector at critical points of f





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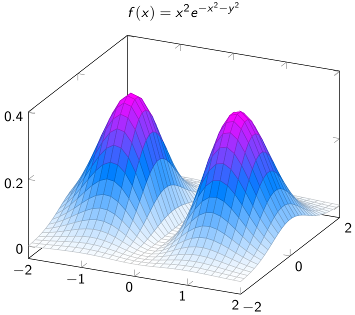
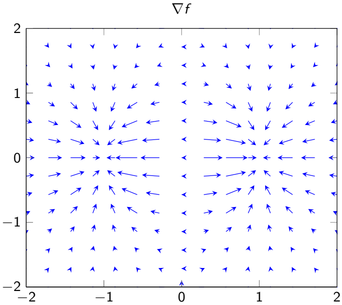
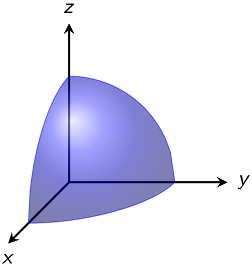




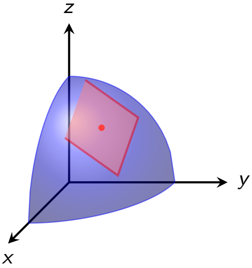
The Derivative - Calculus II



$f(x, y) = \sqrt{4 - x^2 - y^2}$

The gradient vector also gives us a *linear approximation* to the function f near $(x, y) = (a, b)$:

The Derivative - Calculus II



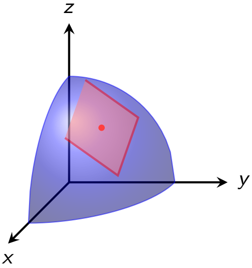
$$f(x, y) = \sqrt{4 - x^2 - y^2}$$

$$L(x, y) = \sqrt{2} - \frac{1}{\sqrt{2}}(x - 1) - \frac{1}{\sqrt{2}}(y - 1)$$

The gradient vector also gives us a *linear approximation* to the function f near $(x, y) = (a, b)$:

$$L(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

The Derivative - Calculus II



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$$L(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

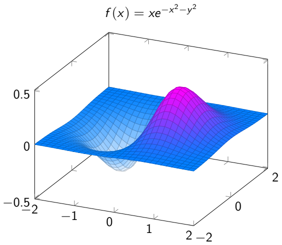
It may help to think of this formula as

$$L(x, y) = f(a, b) + (\nabla f)(a, b) \cdot \langle x - a, y - b \rangle$$

to compare with

$$L(x) = f(a) + f'(a)(x - a)$$

The Second Derivative - Calculus II



If the first derivative is a vector, the second derivative is a *matrix*!

$$(\text{Hess}\, f)(a,b) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial x \partial y}(a,b) \\ \frac{\partial^2 f}{\partial y \partial x}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{pmatrix}$$

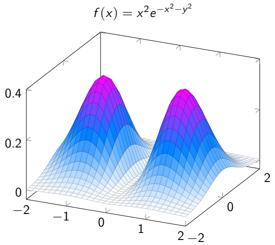
The determinant of the Hessian at a critical point is:

- Positive at a local extremum
- Negative at a saddle

The second derivative $\frac{\partial^2 f}{\partial x^2}(a,b)$ is

- Positive at a *local minimum* of f
- Negative at a *local maximum* of f

The Second Derivative - Calculus II



If the first derivative is a vector, the second derivative is a *matrix*!

$$(\text{Hess}\, f)(a, b) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}(a, b) & \dfrac{\partial^2 f}{\partial x \partial y}(a, b) \\ \\ \dfrac{\partial^2 f}{\partial y \partial x}(a, b) & \dfrac{\partial^2 f}{\partial y^2}(a, b) \end{pmatrix}$$

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The second derivative $\dfrac{\partial^2 f}{\partial x^2}(a, b)$ is

- Positive at a *local minimum* of f
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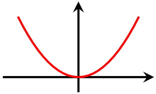
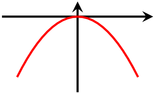
Maxima and Minima in Calculus I and III

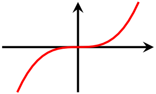
Second Derivative Test - Functions of One Variable



$f(x) = x^2,\ f''(0) > 0$

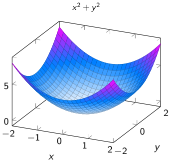


$f(x) = -x^3,\ f''(0) < 0$

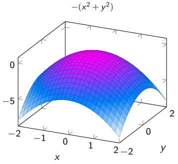


$f(x) = x^3,\ f''(0) = 0$

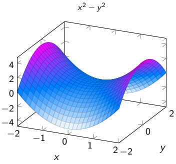
Second Derivative Test - Functions of Two Variables



$f(x,y) = x^2 + y^2,\ D = 4,$
$f_{xx}(0) = 2$



$f(x,y) = -(x^2 + y^2),\ D = 4,$
$f_{xx}(0) = -2$



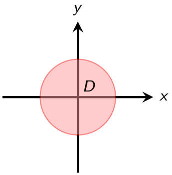
$f(x,y) = x^2 - y^2,\ D = -4$

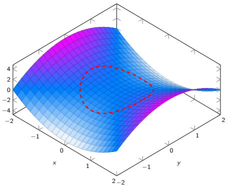
Optimization - Calculus III

To find the absolute maximum and minimum of a function $f(x, y)$ on a domain D:

Example: Optimize the function $f(x, y) = x^2 - y^2$ on the domain

$$D = \{(x, y) : x^2 + y^2 \leq 1\}$$





Optimization - Calculus III

To find the absolute maximum and minimum of a function $f(x, y)$ on a domain D:

- Find the *interior critical points* of f

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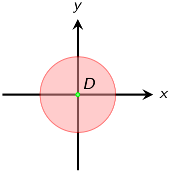
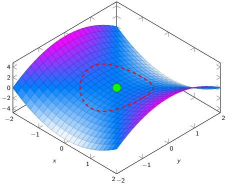




$$\nabla f(x, y) = \langle 2x, -2y \rangle$$

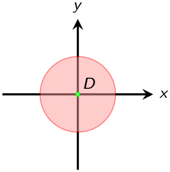
Optimization - Calculus III

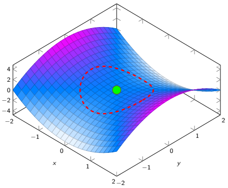
To find the absolute maximum and minimum of a function $f(x, y)$ on a domain D:

- Find the *interior critical points* of f
- Test f at the interior critical points

Example: Optimize the function $f(x, y) = x^2 - y^2$ on the domain

$$D = \{(x, y) : x^2 + y^2 \leq 1\}$$





$\nabla f(x, y) = \langle 2x, -2y \rangle$

$f(0, 0) = 0$

Optimization - Calculus III

To find the absolute maximum and minimum of a function $f(x, y)$ on a domain D:

- Find the *interior critical points* of f
- Test f at the interior critical points
- Use one-variable optimization to find the maximum and minimum of f on each component of the boundary

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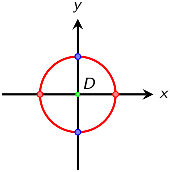
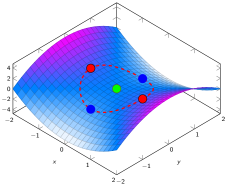
Example: Optimize the function $f(x, y) = x^2 - y^2$ on the domain

$$D = \{(x, y) : x^2 + y^2 \leq 1\}$$





$\nabla f(x, y) = \langle 2x, -2y \rangle$

$f(0, 0) = 0$

Parameterize circle:

$x(t) = \cos(t),\ y(t) = \sin(t)$
$g(t) = \cos^2 t - \sin^2 t$
$g'(t) = -4\cos(t)\sin(t)$

Optimization - Calculus III

To find the absolute maximum and minimum of a function $f(x,y)$ on a domain D:

- Find the *interior critical points* of f
- Test f at the interior critical points
- Use one-variable optimization to find the maximum and minimum of f on each component of the boundary
- The largest value of f in this list is its absolute maximum, and the smallest value of f in this list is its absolute minimum

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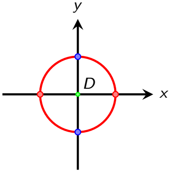
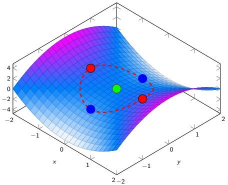
Example: Optimize the function $f(x,y) = x^2 - y^2$ on the domain

$$D = \{(x,y) : x^2 + y^2 \leq 1\}$$





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$g(0) = g(\pi) = 1$
$g(\pi/2) = g(3\pi/2) = -1$