Calculus I Meets Calculus III, Part II: How to Compute Almost Any Integral

Peter A. Perry

University of Kentucky

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Homework

- Finish Homework D3 due Tonight
- Review for Final Exam on Thursday, December 13, 6:00-8:00 PM
- Be sure you know which room to go to for the final!

Final Exam Room Assignments

Sections 001-008	BE 111
Sections 009-013	**BS 107**
Sections 014-016	KAS 213

The Big Picture

Unit I	Vectors and Space Curves
Unit II	Differential Calculus
Unit III	Double and Triple Integrals
Unit IV	Calculus of Vector Fields

Goals of the Day

- Remember the Big Lessons of Calculus I
- Remember all the Integrals from Calculus III
- Review how to Compute Them

Today we'll mainly talk about *integral* calculus

Fundamental Theorems

Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

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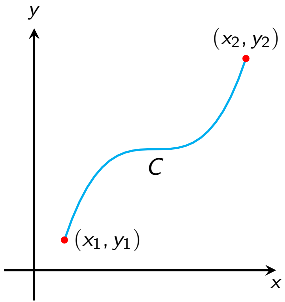

Fundamental Theorem of Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$$

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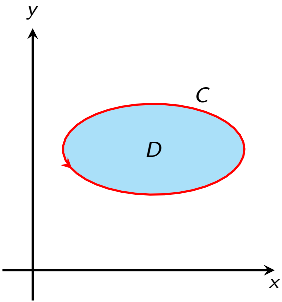


Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA = \oint_C P\,dx + Q\,dy$$

Fundamental Mantra

Fundamental Mantra of Integral Calculus Anything that can be *approximated* as a Riemann sum can be *computed* as a Riemann integral

Area under the graph of $f(x)$	$\int_a^b f(x)\,dx$
Area between graphs of f and g	$\int_a^b f(x) - g(x)\,dx$
Volumes by Washers	$V = \int \pi\left[f(x)^2 - g(x)^2\right]\,dx$
Volumes by Shells	$V = \int 2\pi x\, f(x)\,dx$
Arc length of $y = f(x)$	$\int_a^b \sqrt{1 + f'(x)^2}\,dx$
Arc length $x = x(t), y = y(t)$	$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$

All The Integrals In Calculus III

Group I - Iterated Integrals

Volume under $z = (x, y)$ over D	$\int\int_D f(x,y)\, dA$
Integral of $f(x,y,z)$ over E	$\int\int\int_E f(x,y,z)\, dV$

Group II - Parameterized Integrals

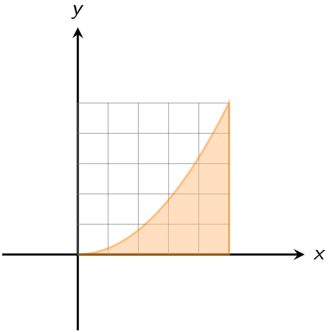
Integral of $f(x,y)$ over C	$\int_C f(x,y)\, ds,$
	$\int_C f(x,y)\, dx,\ \int_C f(x,y) dy$
Integral of $f(x,y,z)$ over C	$\int_C f(x,y,z)\, ds$
Integral of $\mathbf{F}(x,y,z)$ over C	$\int_C \mathbf{F}(x,y,z) \cdot d\mathbf{r}$

Group I: Iterated Integrals

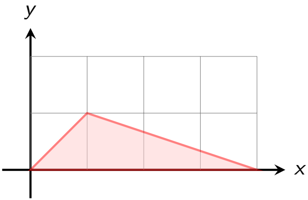
Techniques:

- Iterated integrals for Type I and Type II regions in the *xy* plane
- Polar coordinates for double integrals
- Iterated integrals for Type I, Type II, and Type III regions in *xyz* space
- Cylindrical and Spherical Coordinates
- Change of Variables Theorem

Type I and Type II regions in the xy plane

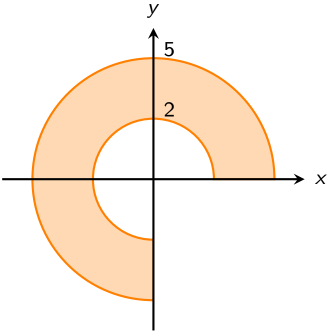


Find $\iint_D x\cos(y)\,dA$ if D is bounded by $y=0,\ y=x^2,\ x=1$



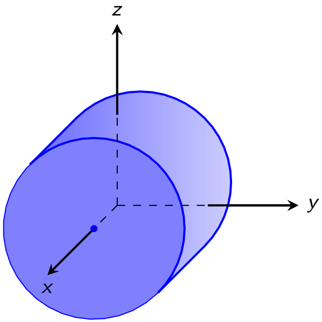
Find $\iint_D y\,dA$ if D is the triangle with vertices $(0,0)$, $(1,1)$, and $(4,0)$

Polar Coordinates



Write $\iint_D f(x, y)\, dA$ as an iterated integral in polar coordinates if D is the region shown at left.

Type I, II and II regions in *xyz* space



Express the integral $\iiint_E f(x, y, z)\, dV$ in three different ways if E is the solid bounded by

$$y^2 + z^2 = 9,$$

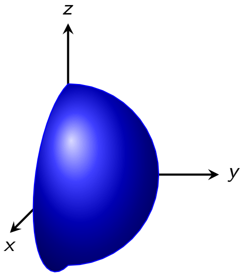
$$x = -2, \text{ and } x = 2$$

Cylindrical and Spherical Coordinates

Graphic will be drawn manually!

Find $\iiint_E x^2\, dV$ if E is the solid that lies:

- in the cylinder $x^2 + y^2 = 1$,
- above the plane $z = 0$, and
- below the cone $z^2 = 4x^2 + 4y^2$

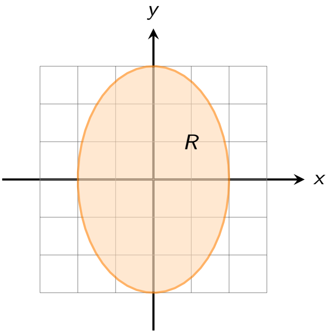


Find $\iiint_E y^2\, dV$ if E is the solid hemisphere $x^2 + y^2 + z^2 = 9$, $y \geq 0$.

Change of Variables Theorem

Recall if $T : S \to D$ then

$$\iint_D f(x,y)\,dA = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$



Find $\iint_R x^2\,dA$ if R is the region bounded by the ellipse

$$9x^2 + 4y^2 = 36.$$

Use the change of variables

$$x = 2u, \quad y = 3v.$$

Group II: Parameterized Integrals

$$\int_C f(x,y)\,ds, \quad \int_C f(x,y)\,dx, \quad \int_C f(x,y)\,dy, \quad \int_C \mathbf{F}(x,y)\cdot d\mathbf{r},$$
$$\int f(x,y,z)\,ds, \quad \int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r}$$

C is always a *parameterized curve*:

$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$,	$a \leq t \leq b$	in two dimensions
$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$,	$a \leq t \leq b$	in three dimensions

Evaluate f or $\mathbf{F}$ along the curve and use

$$dx = x'(t)\,dt, \quad dy = y'(t)\,dt, \quad dz = z'(t)\,dt$$
$$ds = \sqrt{x'(t)^2 + y'(t)^2}\,dt \text{ in two dimensions}$$
$$ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt \text{ in three dimensions}$$

Examples - Integrals in the Plane

Green's Theorem Suppose that C is a positively oriented, piecewise smooth curve surrounding a region D, and that P and Q have continuous partial derivatives on an open region that contains D. Then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

1. Find $\int_C y\,ds$ if C is given by $x = t^2$, $y = 2t$, $0 \leq t \leq 3$
2. Use Green's theorem to find $\int_C (x^2 + y^2)\,dx + (x^2 - y^2)\,dy$ if C is the triangle with vertices $(0, 0)$, $(2, 1)$, and $(0, 1)$.

Examples - Integrals in xyz Space

1. Find $\int_C xye^{yz}\,ds$ if C is the line segment from $(0,0,0)$ to $(1,2,3)$
2. Find $\int_C y\,dx + z\,dy + x\,dz$ if C is the curve $x = \sqrt{t}$, $y = t$, $z = t^2$, $1 \leq t \leq 4$