Math 213 - Vector-Valued Functions

Peter A. Perry

University of Kentucky

September 10, 2018

Homework

- Webwork A4 on section 12.6 is due Wednesday!
- Wednesday is Drop Day
- Quiz # 3 on 12.5–12.6 on Thursday
- Re-re-read section 13.1, pp. 848–853
- Begin working on pp. 853–855, problems 1-13 (odd), 17, 19, 21-26, 31, 43, 45, 49
- Read section 13.2, 855–859

Unit I: Geometry and Motion in Space

Goals of the Day

- Understand what a **vector-valued function** is
- Understand limits and continuity for vector-valued functions
- Learn to visualize space curves:

  (i) by computing their projections onto the *xy*, *xz*, and *yz* planes,

  (ii) by viewing them as intersections of surfaces

Vector-Valued Functions

A **vector-valued function** is a function $\mathbf{r}(t)$ whose *domain* is a set of real numbers and whose *range* is a set of vectors in two- or three-dimensional space. We can specify $\mathbf{r}(t)$ through its *component functions*:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Example you already know: If $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{v} = \langle a, b, c \rangle$, then

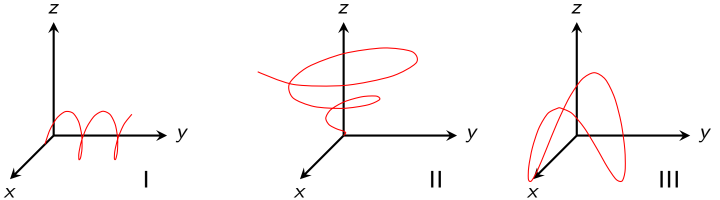
$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

is a vector-valued function with component functions

$$f(t) = x_0 + at, \quad g(t) = y_0 + bt, \quad h(t) = z_0 + ct$$

Vector Function Basics

1. What is the domain of the function $\mathbf{r}(t) = \left\langle \ln(t+1), \frac{t}{9-t^2}, 2^t \right\rangle$?
2. What is $\lim_{t\to 1} \left(\frac{t^2-t}{t-1}\mathbf{i} + \sqrt{t+8}\,\mathbf{j} + \frac{\sin \pi t}{\ln t}\mathbf{k} \right)$?
3. Can you match these curves with their graphs?
   (a) $\mathbf{r}(t) = \langle t\sin t, t\cos t, t \rangle$
   (b) $x = \cos t, y = \sin t, z = \cos 2t$
   (c) $x = \cos t, y = t, z = \sin t$



Breaking it Down: Limits and Continuity

The limit of a vector-valued function is the limit of the component functions:

$$\lim_{t\to t_0}\langle x(t), y(t), z(t)\rangle = \left\langle \lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t), \lim_{t\to t_0} z(t)\right\rangle$$

A vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is continuous at $t = a$ if each of the component functions $x(t)$, $y(t)$, $z(t)$ is continuous at $t = a$

In short, $\mathbf{r}(t)$ is continuous at a if $\lim_{t\to a}\mathbf{r}(t) = \mathbf{r}(a)$

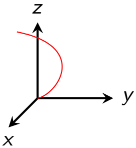
Space Curves

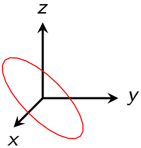
If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a vector function defined on an interval I, the set of all points $(x(t), y(t), z(t))$ for t in the interval I is called a *space curve* C. The equations $x = f(t), y = g(t), z = h(t)$ are called the parametric equations for C.

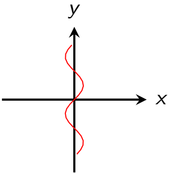
Match each of the space curves shown with their parametric equations.

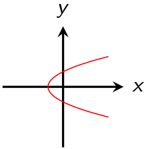
$\mathbf{r}(t) = \langle \sin t, t \rangle$ $\quad$ $\mathbf{r}(t) = \langle t^2 - 1, t \rangle$

$\mathbf{r}(t) = \langle t^2, t^3, t^4 \rangle$ $\quad$ $\mathbf{r}(t) = \langle \cos t, -\cos t, \sin t \rangle$





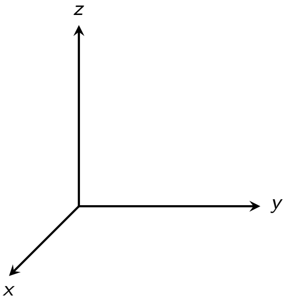




Line Segments are Space Curves

1. Find the vector equation and parametric equations for the line segment from $P(2, 0, 0)$ to $Q(6, 2, -2)$
2. Find the vector equation and parametric equations for the line segment from $P(a, b, c)$ to $Q(u, v, w)$

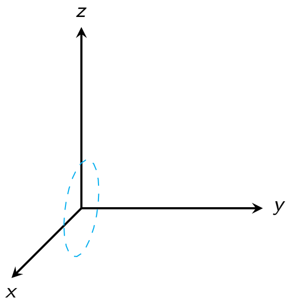
Visualizing It: Projections



Consider the space curve with parametric equations

$$x(t) = \cos t, \quad y(t) = t \quad z(t) = \sin t$$

Visualizing It: Projections

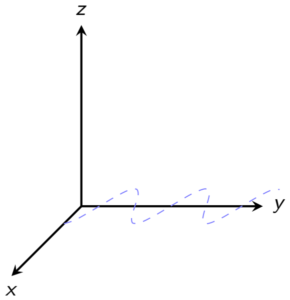


Consider the space curve with parametric equations

$$x(t) = \cos t, \quad y(t) = t \quad z(t) = \sin t$$

1. Find the projection of this curve onto the xz plane (the side wall)

Visualizing It: Projections

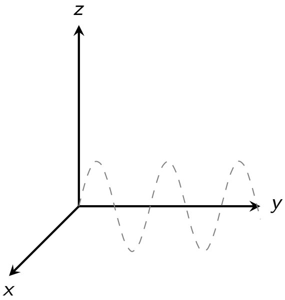


Consider the space curve with parametric equations

$$x(t) = \cos t, \quad y(t) = t \quad z(t) = \sin t$$

1. Find the projection of this curve onto the xz plane (the side wall)
2. Find the projection of this curve onto the xy plane (the floor)

Visualizing It: Projections

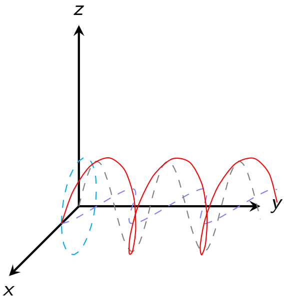


Consider the space curve with parametric equations

$$x(t) = \cos t, \quad y(t) = t \quad z(t) = \sin t$$

1. Find the projection of this curve onto the xz plane (the side wall)
2. Find the projection of this curve onto the xy plane (the floor)
3. Find the projection of this curve onto the yz plane (the back wall)

Visualizing It: Projections

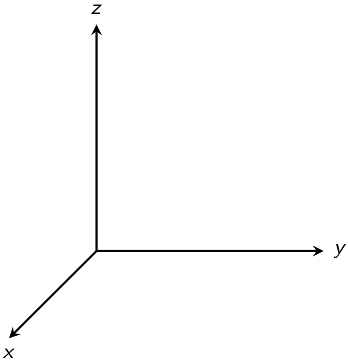


Consider the space curve with parametric equations

$$x(t) = \cos t, \quad y(t) = t \quad z(t) = \sin t$$

1. Find the projection of this curve onto the xz plane (the side wall)
2. Find the projection of this curve onto the xy plane (the floor)
3. Find the projection of this curve onto the yz plane (the back wall)

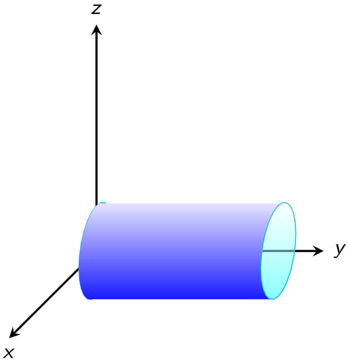
Visualizing It: Surfaces



Let's take another look at the curve

$$x(t) = \cos t, \quad y(t) = t \quad z(t) = \sin t$$

Visualizing It: Surfaces

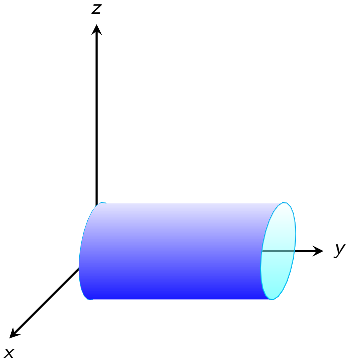


Let's take another look at the curve

$$x(t) = \cos t, \quad y(t) = t \quad z(t) = \sin t$$

1. Show that this curve lies on the cylinder $x^2 + z^2 = 1$

Visualizing It: Surfaces

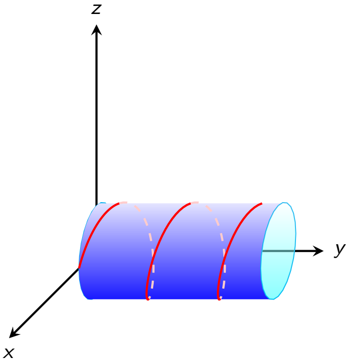


Let's take another look at the curve

$$x(t) = \cos t, \quad y(t) = t \quad z(t) = \sin t$$

1. Show that this curve lies on the cylinder $x^2 + z^2 = 1$
2. Sketch the curve and the surface on the same set of axes

Visualizing It: Surfaces

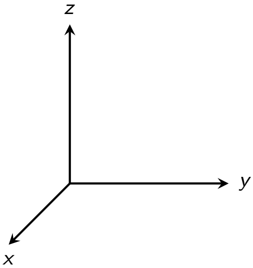


Let's take another look at the curve

$$x(t) = \cos t, \quad y(t) = t \quad z(t) = \sin t$$

1. Show that this curve lies on the cylinder $x^2 + z^2 = 1$
2. Sketch the curve and the surface on the same set of axes

Visualizing It: Surfaces

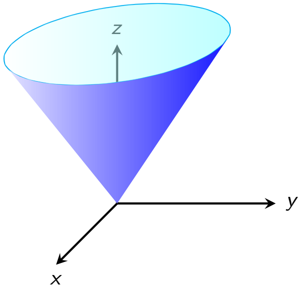


Let's revisit the curve

$$x(t) = t\cos t, \quad y(t) = t\sin t, \quad z(t) = t$$

1. Show that this curve lies on the right circular cone $z^2 = x^2 + y^2$
2. Sketch the curve and the surface on the same set of axes.

Visualizing It: Surfaces

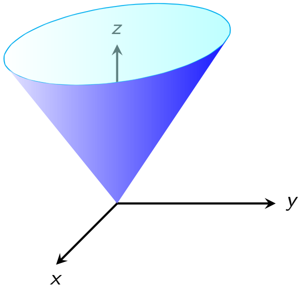


Let's revisit the curve

$$x(t) = t\cos t, \quad y(t) = t\sin t, \quad z(t) = t$$

1. Show that this curve lies on the right circular cone $z^2 = x^2 + y^2$
2. Sketch the curve and the surface on the same set of axes.

Visualizing It: Surfaces

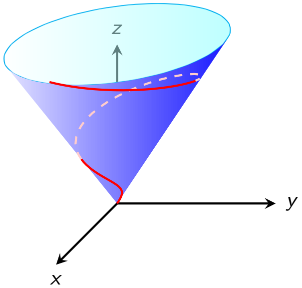


Let's revisit the curve

$$x(t) = t\cos t, \quad y(t) = t\sin t, \quad z(t) = t$$

1. Show that this curve lies on the right circular cone $z^2 = x^2 + y^2$
2. Sketch the curve and the surface on the same set of axes.

Visualizing It: Surfaces



Let's revisit the curve

$$x(t) = t\cos t, \quad y(t) = t\sin t, \quad z(t) = t$$

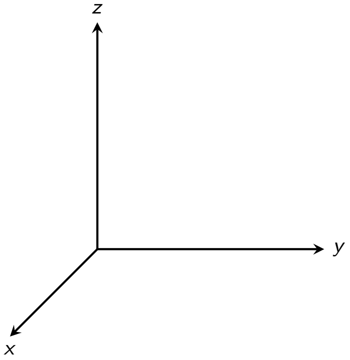
1. Show that this curve lies on the right circular cone $z^2 = x^2 + y^2$
2. Sketch the curve and the surface on the same set of axes.

Visualize It: Space Curves and Surfaces

Show that the curve with parametric equations

$$x = \sin t, \quad y = \cos t, \quad z = \sin^2 t$$

is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$.



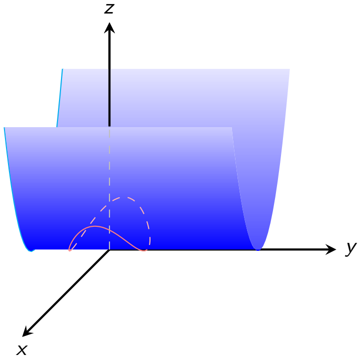
- The surface $z = x^2$ is a cylinder with curve $z = x^2$ parallel to the y-axis
- The surface $x^2 + y^2 = 1$ is a cylinder of radius 1 parallel to the the z-axis
- Their intersection is the parametric curve above

Visualize It: Space Curves and Surfaces

Show that the curve with parametric equations

$$x = \sin t, \quad y = \cos t, \quad z = \sin^2 t$$

is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$.



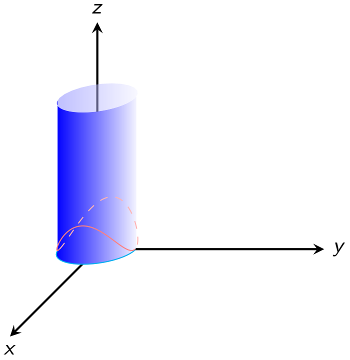
- The surface $z = x^2$ is a cylinder with curve $z = x^2$ parallel to the y-axis
- The surface $x^2 + y^2 = 1$ is a cylinder of radius 1 parallel to the the z-axis
- Their intersection is the parametric curve above

Visualize It: Space Curves and Surfaces

Show that the curve with parametric equations

$$x = \sin t, \quad y = \cos t, \quad z = \sin^2 t$$

is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$.



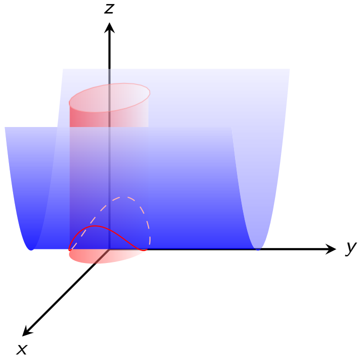
- The surface $z = x^2$ is a cylinder with curve $z = x^2$ parallel to the y-axis
- The surface $x^2 + y^2 = 1$ is a cylinder of radius 1 parallel to the the z-axis
- Their intersection is the parametric curve above

Visualize It: Space Curves and Surfaces

Show that the curve with parametric equations

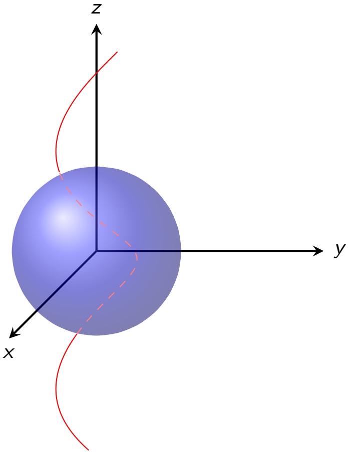
$$x = \sin t, \quad y = \cos t, \quad z = \sin^2 t$$

is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$.



- The surface $z = x^2$ is a cylinder with curve $z = x^2$ parallel to the y-axis
- The surface $x^2 + y^2 = 1$ is a cylinder of radius 1 parallel to the the z-axis
- Their intersection is the parametric curve above

Intersections



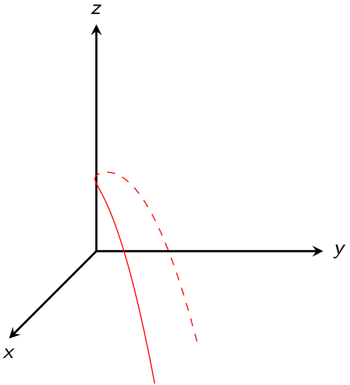
1. Find the points where the helix

$$\mathbf{r}(t) = \langle \sin(t), \cos(t), t \rangle$$

intersects the sphere

$$x^2 + y^2 + z^2 = 5$$

Intersections



1. Find the points where the helix

$$\mathbf{r}(t) = \langle \sin(t), \cos(t), t \rangle$$

intersects the sphere

$$x^2 + y^2 + z^2 = 5$$

2. Find the curve that describes the intersection of the parabolic cylinder $y = x^2$ and the top half of the ellipsoid

$$x^2 + 4y^2 + 4z^2 = 16$$

Intersections

Two particles travel along the space curves

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle, \quad \mathbf{r}_2(t) = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle$$

Do the particles collide? Do their paths intersect?

Recall:

Two particles *collide* if $\mathbf{r}_1(t) = \mathbf{r}_2(t)$ for the *same* t.

Two particles *intersect* if $\mathbf{r}_1(s) = \mathbf{r}_2(t)$ for (possibly different) times s and t.