MA 213 — Calculus III Fall 2017
Exam 2 October 18, 2017

Name: KEY

Section: ____________

Last 4 digits of student ID #: ______

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- All questions are free response questions. Show all your work on the page of the problem. Clearly indicate your answer and the reasoning used to arrive at that answer. Unsupported answers may not receive credit.

Exam Scores

Do not write in the table below

Question	Score	Total
1		8
2		9
3		8
4		8
5		9
6		9
7		9
8		9
9		10
10		10
11		11
Total		100

Free Response. Show your work!

1. (8 points) Use polar coordinates to find the limit:

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}.$$

$$\lim_{(x,y)\to(0,0)} \frac{e^{-(x^2+y^2)}-1}{x^2+y^2} = \lim_{r\to 0} \frac{e^{-r^2}-1}{r^2} \quad \text{L'Hospital}$$

$$= \lim_{r\to 0} \frac{-\cancel{2r}\, e^{-r^2}}{\cancel{2r}}$$

$$= \lim_{r\to 0} -e^{-r^2}$$

$$= -1$$

2. (9 points) Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$ if

$$e^z = xyz.$$

1) $$e^z \frac{\partial z}{\partial x} = yz + xy\frac{\partial z}{\partial x}$$

$$(e^z - xy)\frac{\partial z}{\partial x} = yz$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}}$$

2) $$e^z \frac{\partial z}{\partial y} = xz + xy\frac{\partial z}{\partial y}$$

$$(e^z - xy)\frac{\partial z}{\partial y} = xz$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}}$$

Free Response. Show your work!

3. (8 points) Find an equation for the tangent plane to the surface $z = e^{x-y}$ at the point $(2, 2, 1)$. Write the equation in the form $z = ax + by + d$.

$\frac{\partial z}{\partial x} = e^{x-y}$ $\qquad$ $\frac{\partial z}{\partial y} = -e^{x-y}$ $\qquad$ $z = 1$ at $(2,2)$

$\frac{\partial z}{\partial x}(2,2) = 1$ $\qquad$ $\frac{\partial z}{\partial y}(2,2) = -1$

$$z - 1 = 1(x-2) - 1\cdot(y-2)$$

$$z = 1 + (x-2) - (y-2)$$

$$\boxed{z = 1 + x - y}$$

4. (8 points) Given that $f(x, y)$ is a differentiable function with

$$f(2,5) = 6, \quad f_x(2,5) = 1, \quad f_y(2,5) = -1,$$

use a linear approximation to estimate $f(2.2, 4.9)$.

$\cancel{f(2,2)}$

$$f(2.2, 4.9) \approx f(2,5) + f_x(2,5)\cdot 0.2 + f_y(2,5)(-0.1)$$

$$= 6 + 1(0.2) + (-1)(-0.1)$$

$$= 6 + 0.2 + 0.1$$

$$= 6.3$$

Free Response. Show your work!

5. (9 points) If

$$w = xe^{y/z}, \quad x = t^2, \quad y = 1 - t, \quad z = 1 + 2t,$$

find $\partial w / \partial t$ at the point where $t = 0$.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial x} = e^{y/z} \qquad \frac{\partial w}{\partial y} = \frac{x}{z} e^{y/z} \qquad \frac{\partial w}{\partial z} = -\frac{x}{z^2} e^{y/z}$$

At $t=0$: $x=0$ $y=1$ $z=1$

$$\frac{\partial x}{\partial t} = 2t = 0 \qquad \frac{\partial y}{\partial t} = -1 \qquad \frac{\partial z}{\partial t} = 2$$

$$\therefore \frac{\partial w}{\partial t} = e^{1/1}\cdot(0) + 0\cdot(-1) + 0\cdot 2 = \boxed{0}$$

6. (9 points) Find the directional derivative of the function $f(x, y, z) = x^2y + y^2z$ at the point $(1, 2, 3)$ in the direction of the vector $\mathbf{v} = \langle 2, -1, 2\rangle$.

$$\nabla f = \langle 2xy, x^2 + 2yz, y^2 \rangle$$

$$\nabla f(1,2,3) = \langle 4, 13, 9 \rangle$$

$$\|\vec{v}\| = \sqrt{4+1+4} = 3$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \left\langle \frac{2}{3}, \frac{-1}{3}, \frac{2}{3} \right\rangle$$

$$D_{\vec{u}} f(1,2,3) = \langle 4, 13, 9\rangle \cdot \left\langle \frac{2}{3}, \frac{-1}{3}, \frac{2}{3} \right\rangle$$

$$= \frac{8}{3} - \frac{13}{3} + \frac{18}{3} = \boxed{\frac{13}{3}}$$

7. (9 points) Find the maximal rate of change of $f(x,y) = 4y\sqrt{x}$ at the point $(4,1)$ and the direction in which it occurs.

$\vec{\nabla} f(x,y) = \cancel{\langle 4\sqrt{x},} \langle 2y/\sqrt{x}, 4\sqrt{x} \rangle$

$\vec{\nabla} f(4,1) = \langle 1, 8 \rangle$

$\|\vec{\nabla} f(4,1)\| = \sqrt{1+8^2} = \boxed{\sqrt{65}}$ max. rate of change

Direction $\vec{u} = \frac{1}{\sqrt{65}} \langle 1, 8 \rangle = \boxed{\langle \frac{1}{\sqrt{65}}, \frac{8}{\sqrt{65}} \rangle}$

8. (9 points) Find the parametric equations for the normal line to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

at the point $(-2, 1, -3)$.

If $f(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$, $\nabla f(-2,1,-3)$ points along normal line.

$\nabla f(x,y,z) = \langle \frac{x}{2}, 2y, \frac{2z}{9} \rangle$

$\nabla f(-2,1,-3) = \langle -1, 2, -\frac{2}{3} \rangle$

Parametric eq'n: $\vec{r}(t) = (-2, 1, -3) + t \langle -1, 2, \frac{-2}{3} \rangle$

or: $x(t) = -2 - t$

$y(t) = 1 + 2t$

$z(t) = -3 - \frac{2t}{3}$

9. (10 points) Find the critical points of $f(x,y) = x^4 - 2x^2 + y^3 - 3y$ and classify them as local maximum, local minimum, or saddle point.

$f_x(x,y) = 4x^3 - 4x = 4x(x^2-1)$

$f_y(x,y) = 3y^2 - 3 = 3(y^2-1)$

$\therefore \quad x = 0, \pm 1 \qquad y = \pm 1$

$f_{xx}(x,y) = 12x^2 - 4 \qquad f_{xy} = 0 \qquad f_{yy} = 6y$

$\text{Hess}(f)(x,y) = \begin{pmatrix} 12x^2-4 & 0 \\ 0 & 6y \end{pmatrix}$

$D = (12x^2-4)(6y)$

$(0,1)$: $D = -24$ saddle

$(0,-1)$ $D = 24$ $f_{xx}(0,-1) = -4$ max

$(1,1)$ $D = 48$ $f_{xx}(1,1) = 8$ min

$(1,-1)$ $D = -48$ saddle

$(-1,1)$ $D = 48$ $f_{xx}(-1,1) = 8$ min

$(-1,-1)$ $D = -48$ saddle

10. (10 points) Find the absolute maximum and minimum values of

$$f(x, y) = x + \sqrt{3}\, y$$

on the unit disk

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

and the points of D where these values are reached.

Interior CP $\quad f_x = 1 \quad\quad \Rightarrow$ no interior CP

$\qquad\qquad\quad f_y = \sqrt{3}$

Parameterize bdy: $x(t) = \cos(t) \quad y(t) = \sin t$

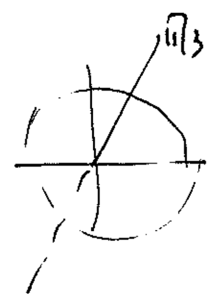
$\varphi(t) = f(x(t), y(t)) = \cos t + \sqrt{3} \sin t$

$\varphi'(t) = -\sin t + \sqrt{3} \cos t$

$\varphi'(t) = 0$ if $\sqrt{3} \cos t = \sin t$

i.e. $\sqrt{3} = \tan t$

i.e. $t = \pi/3,\ 4\pi/3$



$\varphi(\pi/3) = \cos \pi/3 + \sqrt{3} \sin \pi/3 = \frac{1}{2} + \sqrt{3} \cdot \frac{\sqrt{3}}{2} = \frac{1}{2} + \frac{3}{2} = \boxed{2}$

$\varphi(4\pi/3) = \cos(4\pi/3) + \sqrt{3} \sin(4\pi/3) = -\frac{1}{2} + \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{-2}$

Global max! $\boxed{2}$ at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

global min! $\boxed{-2}$ at $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Free Response. Show your work!

11. (11 points) Use Lagrange multipliers to find the extreme values of

$$f(x, y, z) = xy^2z$$

subject to the constraint

$$x^2 + y^2 + z^2 = 4.$$

[*Note*: No credit will be given if a different method is used.]

$f(x,y,z) = xy^2z$ $\qquad$ $g(x,y,z) = x^2+y^2+z^2$

$\nabla f = \langle y^2z,\ 2xyz,\ xy^2\rangle$ $\qquad$ $\nabla g(x,y,z) = \langle 2x, 2y, 2z\rangle$

$\nabla f = \lambda \nabla g$

(1) $y^2z = 2\lambda x \Rightarrow \lambda = \frac{y^2z}{2x}$ $\qquad$ (4) : $\frac{y^2\cancel{z}}{2x} = x\cancel{z} \Rightarrow 2x^2 = y^2$

(2) $2xyz = 2\lambda y \Rightarrow \lambda = xz$ $\qquad$ (5) $\cancel{x}z = \frac{\cancel{x}y^2}{2z} \Rightarrow 2z^2 = y^2$

(3) $xy^2 = 2\lambda z \Rightarrow \lambda = \frac{xy^2}{2z}$

From (4) $x^2 = y^2/2$ $\qquad$ so using constraint:

" (5) $z^2 = y^2/2$

$x^2+y^2+z^2=4$

$y^2/2 + y^2 + y^2/2 = 4$

$2y^2 = 4$

$y^2 = 2$

$y = \pm\sqrt{2}$

$x^2 = z^2 = 1$ so test

$x = \pm 1,\ y = \pm\sqrt{2},\ z = \pm 1$

x	y	z	xy^2z
1	$\sqrt{2}$	1	2
−1	$\sqrt{2}$	1	−2
1	$-\sqrt{2}$	1	2
−1	$-\sqrt{2}$	+1	−2
1			etc

Max +2

Min −2