Math 213 - Functions of Several Variables

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Reminders

1. Homework A6 on 13.3-13.4 is due **tonight**!
2. Homework B1 on 14.1 is due on **Wednesday**
3. Homework B2 on 14.3 is due on **Friday**
4. You will get Exam 1 back in recitation tomorrow. If you have any questions about grading, please return your papers to your TA with a note explaining your concern by **end of recitation on Tuesday**.

Unit II: Functions of Several Variables

New Kinds of Functions

1. Vector-valued functions $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ ✓
2. Functions of several variables $f(x, y)$, $g(x, y, z)$
3. Transformations

$$(u, v) \to (x(u, v), y(u, v))$$

   and

$$(u, v, w) \to (x(u, v, w), y(u, v, w), z(u, v, w))$$

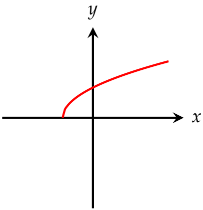
4. Vector Fields

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

Learning Goals

- Know how to find the domain of a function of several variables
- Know how to graph a function of two variables in three-dimensional space
- Know how to find the level curves of a function of two variables and to match the graph of a function with its contour plot
- Know how to find level surfaces of a function of three variables

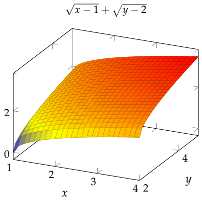
One Variable versus Two Variables



A function of one variable is a map $f : I \to \mathbb{R}$ where the domain, I, is a subset of the real line

Example: $f(x) = \sqrt{1+x}$, $I = [-1, \infty)$

The *graph* of f is the set of points $(x, f(x))$ in the xy plane, where $x \in I$



A function of two variables is a map $f : U \to \mathbb{R}$ where the domain U is a subset of $\mathbb{R}^2$.

Example: $f(x, y) = \sqrt{x-1} + \sqrt{y-2}$,

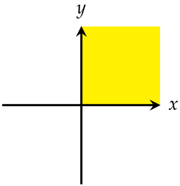
$$U = \{(x, y) : x \geq 1, y \geq 2\}$$

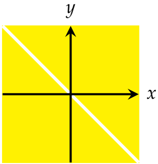
The *graph* of f is the set of points $(x, y, f(x, z))$ in the xyz plane

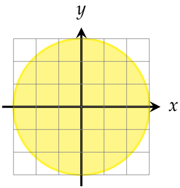
Match the following functions with the graphs of their domains in the xy-plane.

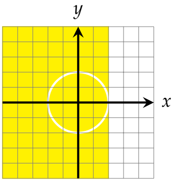
$$f(x,y) = \sqrt{9 - x^2 - y^2} \qquad f(x,y) = \frac{x-y}{x+y}$$

$$f(x,y) = \frac{\ln(2-x)}{4 - x^2 - y^2} \qquad f(x,y) = \sqrt{x} + \sqrt{y}$$







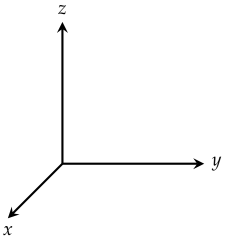


Linear Functions

A function of the form $f(x, y) = ax + bx + c$ for numbers a, b, and c is a *linear function*. Its graph is a plane:

$$z = ax + by + c \Rightarrow ax + by - z = c$$

You already know how to graph this!



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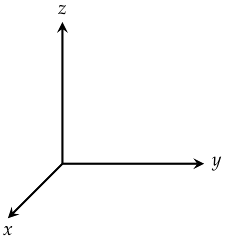

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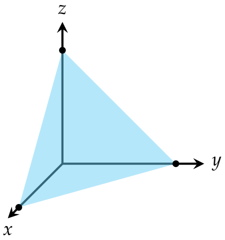

Find the graph of $f(x, y) = 2 - x - y$

$$x + y + z = 2$$

$(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$ all lie on this plane

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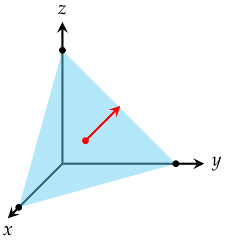

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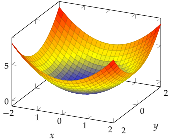
The normal vector is $\langle 1, 1, 1 \rangle$

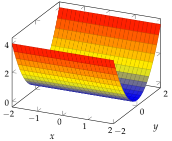
Quadratic Functions

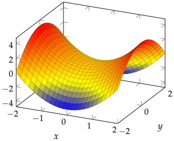
Everything you know about cylinders and quadric surfaces $z = f(x,y)$ tells you something about graphs. Can you match these functions to their graphs?

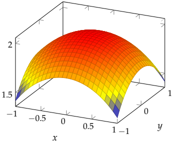
$$f(x,y) = y^2 \qquad f(x,y) = x^2 - y^2$$

$$f(x,y) = \sqrt{4 - x^2 - y^2} \qquad f(x,y) = x^2 + y^2$$







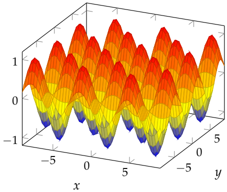


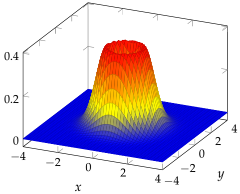
Common Sense and Connection

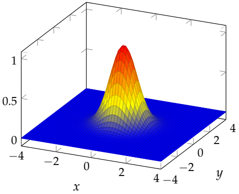
Can you match these functions with their graphs?

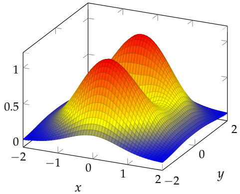
$$f(x,y) = \sin(x)\cos(y) \qquad f(x,y) = \exp(-x^2 - y^2)$$

$$f(x,y) = (x^2 + y^2)e^{-(x^2+y^2)} \qquad f(x,y) = (x^2 + 3y^2)e^{-(x^2+y^2)}$$



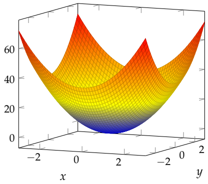






Level Curves

Definition The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant in the range of f.



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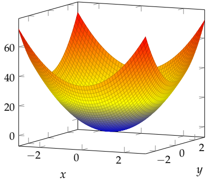

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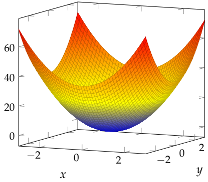
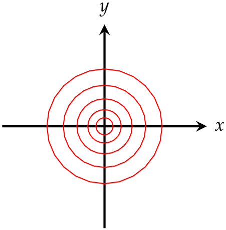


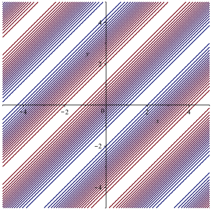


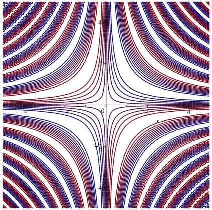
- What is the range of the function $f(x,y) = x^2 + y^2$?
- Describe the level curves of this function

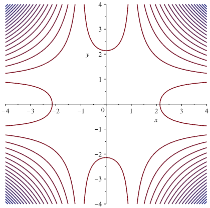
Contour Plots

A **contour plot** of a function shows a number of level curves. Can you match these functions with their graphs and contour plots?

$$f(x,y) = \sin(xy) \qquad f(x,y) = (1-x^2)(1-y^2) \qquad f(x,y) = \sin(x-y)$$







You Already Know About Contour Plots

Let's examine a topo map from the Great Smoky Mountains National Park courtesy of the United States Geological Survey (USGS)

Functions of Three Variables

A function of three variables is a map $f : V \to \mathbb{R}$ where the domain V is a subset of $\mathbb{R}^3$

Find the domain and range of these functions of three variables

1. $f(x, y, z) = x^2 + y^2 + z^2$
2. $f(x, y, z) = \sqrt{9 - x^2 - y^2 - z^2}$
3. $f(x, y, z) = x + y + z$

Definition The **level surfaces** of a function f of three variables are the surfaces with equation $f(x,y,z) = k$ where k is a constant in the range of f.

Determine the level surfaces of the the following functions:

1. $f(x,y,z) = x^2 + y^2 + z^2$
2. $f(x,y,z) = \sqrt{9 - x^2 - y^2 - z^2}$
3. $f(x,y,z) = x + y + z$

Summary

- We learned how to find the domain of a function of two variables and find its graph in three dimensional space
- We learned how to find level curves for a function of two variables and level surfaces for a function of three variables