### Math 213 - Functions of Several Variables

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### Reminders

- 1 Homework A6 on 13.3-13.4 is due **tonight**!
- 2 Homework B1 on 14.1 is due on Wednesday
- 3 Homework B2 on 14.3 is due on Friday
- You will get Exam 1 back in recitation tomorrow. If you have any questions about grading, please return your papers to your TA with a note explaining your concern by end of recitation on Tuesday.

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## Unit II: Functions of Several Variables

13.3-4 Lecture 11: Velocity and Acceleration

- 14.1 Lecture 12: Functions of Several Variables
- 14.3 Lecture 13: Partial Derivatives
- 14.4 Lecture 14: Linear Approximation
- 14.5 Lecture 15: Chain Rule, Implicit Differentiation
- 14.6 Lecture 16: Directional Derivatives and the Gradient
- 14.7 Lecture 17: Maximum and Minimum Values, I
- 14.7 Lecture 18: Maximum and Minimum Values, II
- 14.8 Lecture 19: Lagrange Multipliers
- 15.1 Double Integrals
- 15.2 Double Integrals over General Regions Exam II Review

### New Kinds of Functions

- **1** Vector-valued functions  $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$   $\checkmark$
- **2** Functions of several variables f(x, y), g(x, y, z)
- 3 Transformations

$$(u,v) \to (x(u,v),y(u,v)$$

and

$$(u,v,w) \to (x(u,v,w), y(u,v,w), z(u,v,w))$$

4 Vector Fields

$$\mathbf{F}(x,y,z) = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$$

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## Learning Goals

- Know how to find the domain of a function of several variables
- Know how to graph a function of two variables in three-dimensional space
- Know how to find the level curves of a function of two variables and to match the graph of a function with its contour plot
- Know how to find level surfaces of a function of three variables

### One Variable versus Two Variables



A function of one variable is a map  $f : I \to \mathbb{R}$  where the domain, *I*, is a subset of the real line

Example:  $f(x) = \sqrt{1+x}, I = [-1, \infty)$ 

The *graph* of *f* is the set of points (x, f(x)) in the *xy* plane, where  $x \in I$ 



A function of two variables is a map  $f : U \to \mathbb{R}$ where the domain *U* is a subset of  $\mathbb{R}^2$ .

Example:  $f(x, y) = \sqrt{x - 1} + \sqrt{y - 2}$ ,

$$U = \{(x, y) : x \ge 1, y \ge 2\}$$

The *graph* of *f* is the set of points (x, y, f(x, z)) in the *xyz* plane

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Match the following functions with the graphs of their domains in the *xy*-plane.

$$\begin{aligned} f(x,y) &= \sqrt{9 - x^2 - y^2} & f(x,y) = \frac{x - y}{x + y} \\ f(x,y) &= \frac{\ln(2 - x)}{4 - x^2 - y^2} & f(x,y) = \sqrt{x} + \sqrt{y} \end{aligned}$$



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# Graphing Functions of Two Variables Contour Plots Level Surfaces

### Linear Functions

A function of the form f(x, y) = ax + bx + c for numbers a, b, and c is a *linear function*. Its graph is a plane:

$$z = ax + by + c \Rightarrow ax + by - z = c$$

You already know how to graph this!



# Graphing Functions of Two Variables Contour Plots Level Sur

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x + y + z = 2(2,0,0), (0,2,0), and (0,0,2) all lie on this plane

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### Graphing Functions of Two Variables

Contour Plots

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The normal vector is  $\langle 1, 1, 1 \rangle$ 

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### Graphing Functions of Two Variables Contour Plots Level Surfaces

### **Quadratic Functions**

Everything you know about cylinders and quadric surfaces z = f(x, y) tells you something about graphs. Can you match these functions to their graphs?

$$f(x,y) = y^{2} \qquad f(x,y) = x^{2} - y^{2}$$
$$f(x,y) = \sqrt{4 - x^{2} - y^{2}} \qquad f(x,y) = x^{2} + y^{2}$$





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### Common Sense and Connection

Can you match these functions with their graphs?

$$f(x,y) = \sin(x)\cos(y) \qquad f(x,y) = \exp(-x^2 - y^2)$$
  
$$f(x,y) = (x^2 + y^2)e^{-(x^2 + y^2)} \qquad f(x,y) = (x^2 + 3y^2)e^{-(x^2 + y^2)}$$





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with equations f(x, y) = k, where *k* is a constant in the range of *f*.







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**Definition** The **level curves** of a function *f* of two variables are the curves with equations f(x, y) = k, where *k* is a constant in the range of *f*.



- What is the range of the function  $f(x,y) = x^2 + y^2$ ?
- Describe the level curves of this function

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### **Contour Plots**

A **contour plot** of a function shows a number of level curves. Can you match these functions with their graphs and contour plots?

 $f(x,y) = \sin(xy)$   $f(x,y) = (1-x^2)(1-y^2)$   $f(x,y) = \sin(x-y)$ 





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### You Already Know About Contour Plots

Let's examine a topo map from the Great Smoky Mountains National Park courtesy of the United States Geological Survey (USGS)

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### Functions of Three Variables

A function of three variables is a map  $f : V \to \mathbb{R}$  where the domain *V* is a subset of  $\mathbb{R}^3$ Find the domain and range of these functions of three variables

1  $f(x, y, z) = x^2 + y^2 + z^2$ 2  $f(x, y, z) = \sqrt{9 - x^2 - y^2 - z^2}$ 3 f(x, y, z) = x + y + z

**Definition** The **level surfaces** of a function f of three variables are the surfaces with equation f(x, y, z) = k where k is a constant in the range of f.

Determine the level surfaces of the the following functions:

1 
$$f(x, y, z) = x^2 + y^2 + z^2$$
  
2  $f(x, y, z) = \sqrt{9 - x^2 - y^2 - z^2}$   
3  $f(x, y, z) = x + y + z$ 

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## Summary

- We learned how to find the domain of a function of two variables and find its graph in three dimensional space
- We learned how to find level curves for a function of two variables and level surfaces for a function of three variables