Math 213 - Partial Derivatives

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Reminders

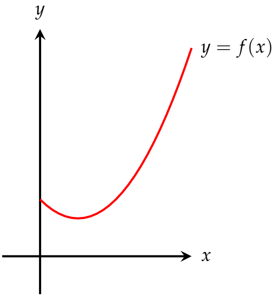
- Homework B1 on 14.1 is due on **tonight**!
- You will have quiz #3 on 13.1-13.4 on **Thursday**
- Homework B2 on 14.3 is due on **Friday**

Unit II: Functions of Several Variables

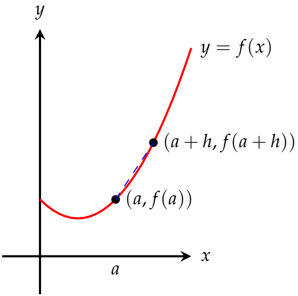
Learning Goals

- Learn how to compute partial derivatives and know various different notations for them
- Understand the geometric interpretation of partial derivatives
- Know how to compute higher partial derivatives
- Understand their connection with partial differential equations

Derivatives - One Variable



Derivatives - One Variable



The derivative of f at a is the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if it exists.

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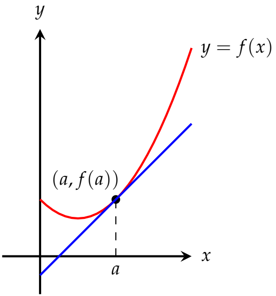

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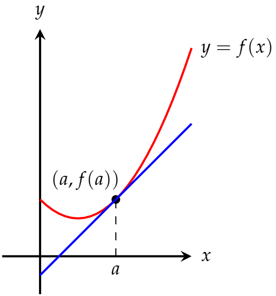

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if it exists.

$f'(a)$ is the slope of the tangent line to the graph of f at the point $(a, f(a))$.

$f'(a)$ is also the instantaneous rate of change of $y = f(x)$ at $x = a$

Partial Derivatives - Two Variables

A function of two variables has two very natural rates of change:

- The rate of change of $z = f(x, y)$ with respect to x when y is fixed
- The rate of change of $z = f(x, y)$ when respect to y when x is fixed

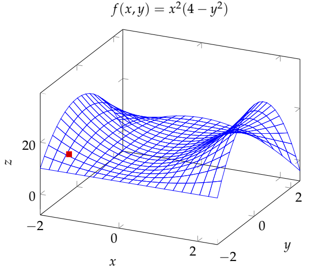
The first of these is called the *partial derivative of f with respect to x*, denoted $\partial f / \partial x$ or f_x

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

the second is called the *partial derivative of f with respect to y*, denoted $\partial f / \partial y$ or f_y

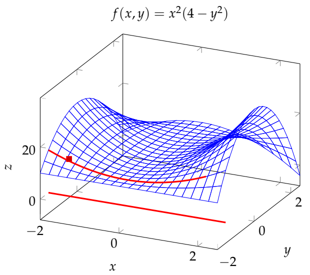
$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

Geometric Interpretation



Given a function $f(x, y)$ ...

Geometric Interpretation

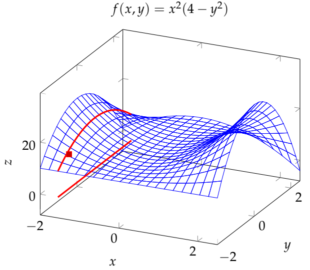


Given a function $f(x, y) \ldots$

Compute $f_x(a, b)$ by setting $y = b$ and varying x:

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

Geometric Interpretation



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$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

Compute $f_y(a,b)$ by setting $x = a$ and varying y:

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Partial Derivatives

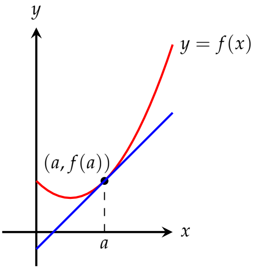
Rules for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y

Find both partial derivatives of the following functions:

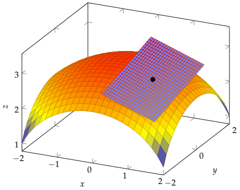
1. $f(x, y) = x^4 + 5xy^3$
2. $f(x, t) = t^2 e^{-x}$
3. $g(u, v) = (u^2 + v^2)^3$
4. $f(x, y) = \sin(xy)$
5. $f(George, Fran) = (George)^5 + (Fran)^3$

Tangent Planes - Sneak Preview



In calculus of one variable, the derivative $f'(a)$ defines a *tangent line* to the graph of f at $(a, f(a))$ by the equation

$$L(x) = f(a) + f'(a)(x - a)$$



In calculus of two variables, the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ define a *tangent plane* to the graph of f at $(a, b, f(a, b))$ by

$$\begin{aligned} L(x, y) &= f(a, b) \\ &\quad + f_x(a, b)(x - a) + f_y(a, b)(y - b) \end{aligned}$$

More Partial Derivatives

Sometimes it's useful to remember that, to compute a partial derivative like $f_x(x, 1)$, you can set $y = 1$ before you start computing.

Find the following partial derivatives.

1. $f_x(x, 1)$ if $f(x, y) = x^{y^{y^{y^{y}}}} \sin(x)$
2. $f_y(3, y)$ if $f(x, y) = (x - 3)\sin(\cos(\log(y)) + xy$

Higher Partials

We can compute higher-order partial derivatives just by repeating operations. We'll find out what these partials actually mean later on!

Example Find the second partial derivatives of $f(x, y) = x^2 y^2$

$$\frac{\partial f}{\partial x} = f_x(x, y) = 2xy^2, \quad \frac{\partial f}{\partial y} = 2x^2 y$$

$$\frac{\partial^2 f}{\partial x^2} = \qquad \frac{\partial^2 f}{\partial x \partial y} =$$

$$\frac{\partial^2 f}{\partial y \partial x} = \qquad \frac{\partial^2 f}{\partial y^2} =$$

Notations:

$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = (f_x)_y, \quad \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = (f_y)_x$$

Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Check Clairaut's theorem for the function $f(x, y) = x^3y^2 - \sin(xy)$

Implicit Differentiation

You can find partial derivatives by implicit differentiation.

1. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $x^2 + y^2 + z^2 = 1$
2. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $e^z = xyz$

Partial Differential Equations

Partial Differential Equations describe many physical phenomena. The unknown function is a function of two or more variables.

The *wave equation* for $u(x,t)$, a function which, for each t gives a 'snapshot' of a one-dimensional traveling wave:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2}$$

The *heat equation* for $u(x,y,t)$, the temperature of a thin sheet at position (x,y) at time t:

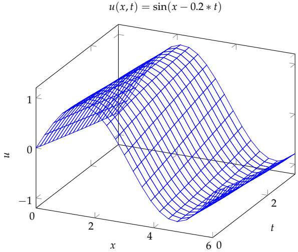
$$\frac{\partial u}{\partial t}(x,y,t) = K\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u(x,y,t)$$

Laplace's Equation for the electrostatic potential of a charge distribution ρ:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u(x,y,z) = 4\pi\rho(x,y,z)$$

The Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2}$$



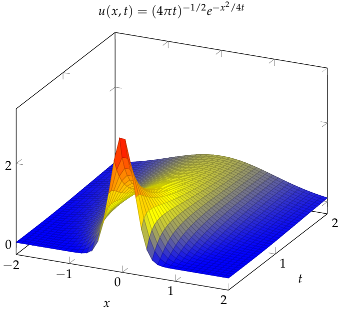
$u(x,t)$ gives the height of a wave moving down a channel as a function of distance x and time t

For each fixed t, we get a "snapshot" of the wave

For each fixed x, we get the height of the wave, at that point, as a function of time

The Heat Equation

$$\frac{\partial u}{\partial t}(x,t) = K\frac{\partial^2}{\partial x^2}u(x,t)$$



For each t we get a "snapshot" of the distribution of heat–at first heat concentrates near $x = 0$, but then diffuses and cools as time moves forward

Summary

- We learned how to compute partial derviatives of functions of several variables and the various notations for them
- We interpreted partial derivatives geometrically
- We learned how to compute higher derivatives
- We saw how partial derivatives arise in equations that describe heat flow and wave propagation