Linear Approximation	

Math 213 - Linear Approximation

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University of Kentucky

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Linear Approximation	

Reminders

• Homework B2 on 14.3 is due tonight!



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Unit II: Functions of Several Variables

13.3-4 Lecture 11: Velocity and Acceleration

- 14.1 Lecture 12: Functions of Several Variables
- 14.3 Lecture 13: Partial Derivatives
- 14.4 Lecture 14: Linear Approximation
- 14.5 Lecture 15: Chain Rule, Implicit Differentiation
- 14.6 Lecture 16: Directional Derivatives and the Gradient
- 14.7 Lecture 17: Maximum and Minimum Values, I
- 14.7 Lecture 18: Maximum and Minimum Values, II
- 14.8 Lecture 19: Lagrange Multipliers
- 15.1 Double Integrals
- 15.2 Double Integrals over General Regions Exam II Review

Tangent Plane Linear Approximation The Differential Three Variables 0000 0000 0 000

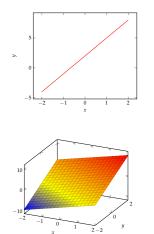
Learning Goals

- Understand how the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ define the *tangent plane* to the graph of z = f(x, y) at (a, b, f(a, b))
- Understand how the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ define the *linear approximation* L(x, y) to f(x, y) near (x, y) = (a, b)
- Understand the *total differential* dz of a function z = f(x, y) and how it's used to compute percentage change and analyze error

• Generalize these ideas to functions of three variables



Warm-Up: Linear Functions



The graph of a line Ax + By = C defines a *linear function* of one variable

$$y = f(x) = \frac{C}{B} - \frac{A}{C}x$$

The graph of a plane ax + by + cz = d defines a *linear function* of two variables

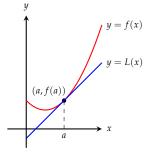
$$z = f(x, y) = \frac{d}{c} - \frac{a}{c}x - \frac{b}{c}y$$

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Functions of One Variable - Tangent Line



The derivative f'(a) gives the slope of the tangent line to the graph of y = f(x) at (a, f(a)).

The derivative f'(a) defines a linear function

$$L(x) = f(a) + f'(a)(x - a)$$

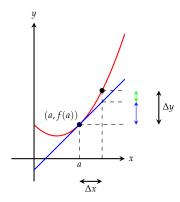
the linear approximation to f near a

The differential of y = f(x) is

 $dy = f'(x) \, dx$

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Functions of One Variable - Differentiability



Recall that if y = f(x), the *increment* of *y* as *x* changes from *a* to $a + \Delta x$ is

 $\Delta y = f(a + \Delta x) - f(a).$

If f is differentiable at a, then

 $\Delta y = f'(a)\,\Delta x + \varepsilon \Delta x$

where

 $\varepsilon \to 0$ as $\Delta x \to 0$

That is, the linear approximation is *very* good as $\Delta x \rightarrow 0$.

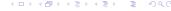
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 Tangent Plane
 Linear Approximation
 The Differential
 Three Variables

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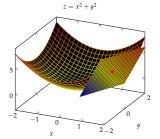
Derivatives - Two Variables

 $z = x^2 + y^2$ $z = x^2 + y^2$ y^2 $z = x^2 + y^2$ y^2 $z = x^2 + y^2$ y^2 $z = x^2 + y^2$ $z = x^2 + y^2$ The derivatives $f_x(a,b)$ and $f_y(a,b)$ define a *tangent plane* to the graph of *f* at (a,b,f(a,b))



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Derivatives - Two Variables



The derivatives $f_x(a,b)$ and $f_y(a,b)$ define a *tangent plane* to the graph of *f* at (a,b,f(a,b))

These derivatives define a linear function

$$L(x,y) = f(a,b)$$

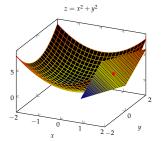
+ $f_x(a,b)(x-a)$
+ $f_y(a,b)(x-b)$

the linear approximation to f near (a, b)

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Derivatives - Two Variables



The derivatives $f_x(a,b)$ and $f_y(a,b)$ define a *tangent plane* to the graph of *f* at (a,b,f(a,b))

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the linear approximation to f near (a, b)

The differential of z = f(x, y) is

$$dz = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy$$

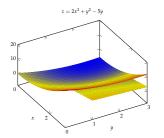
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Find the Tangent Plane

If *f* has continuous partial derivatives, the tangent plane to z = f(x, y) at (a, b, f(a, b)) is

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$



1 Find the equation of the tangent plane to the surface

$$z = 2x^2 + y^2 - 5y$$

at (1, 2, -4).

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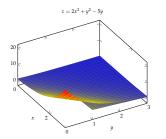
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Find the Tangent Plane

If *f* has continuous partial derivatives, the tangent plane to z = f(x, y) at (a, b, f(a, b)) is

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$



1 Find the equation of the tangent plane to the surface

$$z = 2x^2 + y^2 - 5y$$

2 Find the equation of the tangent plane to the surface

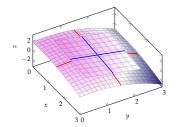
$$z = e^{x-y}$$

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at (2, 2, 1).

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The Tangent Plane Contains Tangent Lines



The red curves represent f(a, y) and f(x, b)

The blue lines are the tangent lines

$$r_1(t) = \langle a, b, f(a, b) \rangle + t \langle 1, 0, f_x(a, b) \rangle$$

$$r_2(t) = \langle a, b, f(a, b) \rangle + t \langle 0, 1, f_y(a, b) \rangle$$

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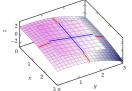
The Tangent Plane Defines a Linear Approximation



The tangent line is the graph of a linear function

L(x) = f(a) + f'(a)(x - a)

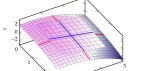
that approximates f(x) near x = a



The tangent plane is the graph of a linear function

L(x,y) = f(a,b) + $f_x(a,b)(x-a) + f_y(a,b)(y-b)$ that approximates f(x, y) near (x, y) = (a, b)

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The Linear Approximation

The linear approximation to f(x, y) at (a, b) is

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

- **1** Show that the linear approximation to $f(x, y) = e^x \cos(xy)$ at (0, 0) is L(x, y) = x + 1
- 2 Suppose that f(2,5) = 6, $f_x(2,5) = 1$, and $f_y(2,5) = -1$. Use a linear approximation to estimate f(2.2, 4.9)

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Linear Approximation	
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Differentiability

If z = f(x, y), the *increment* of z as x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$ is:

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

f is *differentiable* at (a, b) if

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and ε_2 approach 0 as $(\Delta x, \Delta y) \rightarrow (0, 0)$.

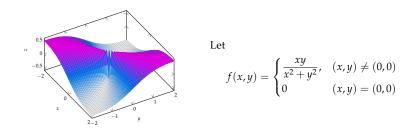
Theorem If the partial derivatives f_x and f_y of f exist near (a, b), and are continuous at (a, b), then f is differentiable at (a, b).

1 Explain why the function $f(x, y) = \sqrt{xy}$ is differentiable at (1,4) and find its linearization

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Linear Approximation	Three Variables
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What Happens if f is not differentiable?



Use the definition to check that $f_x(0,0) = f_y(0,0) = 0$ Show that $f_x(x,y)$ and $f_y(x,y)$ are not continuous at (0,0)

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	Linear Approximation	The Differential		
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Differentials				

For a function *f* of one variable, the differential of y = f(x) is given by

$$dy = f'(x) \, dx$$

For a function *f* of two variables, the differential of z = f(x, y) is

$$dz = f_x(x, y) \, dx + f_y(x, y) \, dy = \frac{\partial z}{\partial x} \, dx + \frac{\partial z}{\partial y} \, dy$$

- 1. The radius of a circle is measured as 10cm with an error of at most 0.2cm. What is the maximum calculated area of the circle?
- 2. The length and width of a rectangle are measured as 30cm and 24cm, with an error of at most 0.1cm each. What is the maximum error in the calculated area of the rectangle?

Linear Approximation		Three Variables	
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Three Variables

If w = f(x, y, z):

• The *linear approximation* of *f* at (*a*, *b*, *c*) is

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - a) + f_z(a, b, c)(z - c)$$

• The *increment* of *w* is

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

• The differential dw is

$$dw = \frac{\partial w}{\partial x} \, dx + \frac{\partial w}{\partial y} \, dy + \frac{\partial w}{\partial z} \, dz$$

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Linear Approximation	Three Variables	
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Three Variables

The *linear approximation* of f at (a, b, c) is

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - a) + f_z(a, b, c)(z - c)$$

Find the linear approximation to

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

at (x, y, z) = (3, 2, 6) and estimate

$$\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$$

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	Linear Approximation	Three Variables
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Summary		

- We showed how the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ determine the tangent plane to the graph of f at (a, b, f(a, b))
- We saw how the tangent plane defines a linear approximation *L*(*x*, *y*) to *f*(*x*, *y*) near (*x*, *y*) = (*a*, *b*)
- We saw how these ideas generalize to functions of three variables