Math 213 - Maximum and Minimum Values, I

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October 4, 2019

Reminders

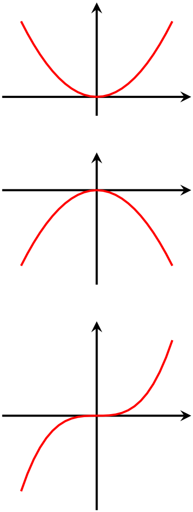
- Homework B4 on 14.5 is due tonight

Unit II: Functions of Several Variables

Learning Goals

- Know how to find a critical point of a function of two variables
- Know how to use the second derivative test to determine whether a given critical point is a local maximum, a local minimum, or a saddle point
- Know how to use the second derivative test to solve simple maximization and minimization problems

Review of Calculus I

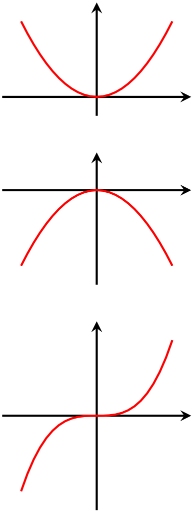


If $y = f(x)$ then local maxima and minima occur at *critical points* a where $f'(a) = 0$ or $f'(a)$ does not exist. There are two main tests:

First Derivative Test:

- If $f'(a) = 0$ and $f'(x)$ changes from $-$ to $+$ then $f(a)$ is a local minimum value
- If $f'(a) = 0$ and $f'(x)$ changes from $+$ to $-$ at $x = a$, then $f(a)$ is a local maximum value
- If $f'(a) = 0$ but $f'(x)$ does not change sign at $x = a$, then $f(a)$ is neither a local maximum value nor a local minimum value

Review of Calculus I

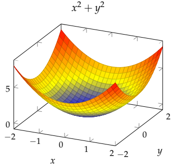


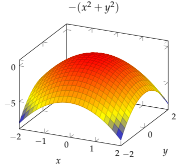
If $y = f(x)$ then local maxima and minima occur at *critical points* a where $f'(a) = 0$ or $f'(a)$ does not exist. There are two main tests:

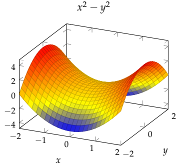
Second Derivative Test:

- If $f'(a) = 0$ and $f''(a) > 0$, then $f(a)$ is a local minimum value
- If $f'(a) = 0$ and $f''(a) < 0$, then $f(a)$ is a local maximum value
- If $f'(a) = 0$ but $f''(a) = 0$, the test is indeterminate

Preview of Calculus III







There is *no* "first derivative test" and instead the nature of the critical point depends on the *Hessian matrix*

$$\text{Hess}(f)(a,b) = \begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{pmatrix},$$

and its determinant

$$D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$$

In the graphs at left:

- $\text{Hess}(f)(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad D = +4$
- $\text{Hess}(f)(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad D = +4$
- $\text{Hess}(f)(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \quad D = -4$

Local and Absolute Extrema

A function $f(x, y)$ has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ for all (x, y) near (a, b).

A function $f(x, y)$ has a local minimum at (a, b) if $f(x, y) \geq f(a, b)$ for all (x, y) near (a, b).

What does it mean for a function to have an absolute maximum value (resp. absolute minimum value) at (a, b)?

Hunting License for Local Extrema

Theorem If f has a local maximum or a local minimum at (a, b), and the first-order partial derivatives of f exist there, then

$$f_x(a, b) = f_y(a, b) = 0.$$

A value of (a, b) where $f_x(a, b)$ and $f_y(a, b)$ are either zero or do not exist is called a *critical point* for the function f.

Critical Points

To find the critical points of a function $f(x, y)$, you need to solve for the values (a, b) that make *both* $f_x(a, b)$ and $f_y(a, b)$ equal to zero.

Examples

1. Find all the critical points of the function $f(x, y) = x^3 - 3x + 3xy^2$
2. Find the critical points of $f(x, y) = x^2 + y^4 + 2xy$
3. Find the critical points of $f(x, y) = e^x \cos y$

Second Derivative Test

Second Derivatives Test Suppose f has second partial derivatives continuous on a disc at (a, b), and $f_x(a, b) = f_y(a, b) = 0$. Let

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum

(c) If $D < 0$, then $f(a, b)$ is a saddle point (neither a maximum nor a minimum)

(a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum
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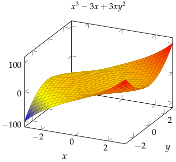
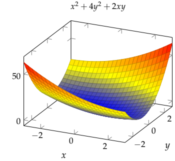
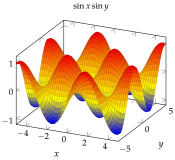
Classify the critical points of the following functions:

1. $f(x,y) = x^3 - 3x + 3xy^2$
2. $f(x,y) = x^2 + y^4 + 2xy$
3. $f(x,y) = \sin(x)\sin(y)$


$x^3 - 3x + 3xy^2$
100
0
−100
−2
0
2
x
−2
0
2
y


$x^2 + 4y^2 + 2xy$
50
0
−2
0
2
x
−2
0
2
y


$\sin x \sin y$
1
0
−1
−4
−2
0
2
4
x
−5
0
5
y

Maximum and Minimum Problems

1. Find the shortest distance from the point $(2,0,-3)$ to the plane $x+y+z=1$.
2. Find the point on the plane $x-2y^3z=6$ that is closest to the point $(0,1,1)$.

Review of Calculus I

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function on a closed interval $[a, b]$:

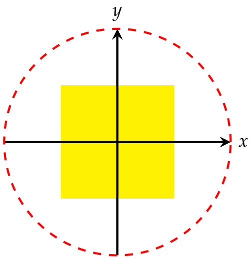
1. Find the values of f at the critical numbers of f in $[a, b]$
2. Find the values of f at the endpoints of the interval
3. The largest of the values from steps 1 and 2 is the absolute maximum of f on $[a, b]$; the smallest of these values is the absolute minimum of f on $[a, b]$.

For functions of two variables:

1. The "closed interval" on the line is replaced by a "closed set" in the plane
2. The boundary of a closed set is a *curve* rather than just two points

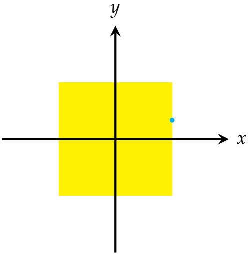
Otherwise, the idea is much the same!

Bounded Sets, Closed Sets, Boundaries



A *bounded set* D in $\mathbb{R}^2$ is a set that can be enclosed inside a large enough circle

Bounded Sets, Closed Sets, Boundaries



A *bounded set* D in $\mathbb{R}^2$ is a set that can be enclosed inside a large enough circle

A *boundary point* is a point (a, b) that has points in D and has points that don't belong to D arbitrarily close to it

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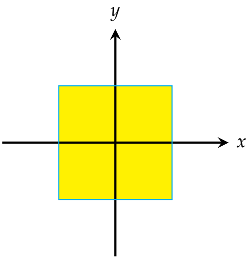

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The *boundary* of a set D is the set consisting of all the boundary points

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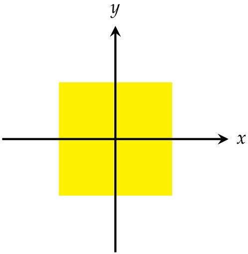

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The *boundary* of a set D is the set consisting of all the boundary points

A *closed* set D is one that contains all of its boundary points.

Bounded Sets, Closed Sets, Boundaries

Classify each of the following sets as bounded or not bounded, and closed or not closed

1. $D = \{(x, y) : x^2 + y^2 < 1\}$
2. $D = \{(x, y,) : x^2 + y^2 \leq 1\}$
3. $D = \{(x, y) : x^2 + y^2 \geq 1\}$
4. $D = \{(x, y) : x^2 + y^2 > 1\}$

The Extreme Value Theorem

Extreme Value Theorem If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Practical fact: These extreme values occur either in the interior of D, where the second derivative test works, or on the boundary of D, where the search for maxima and minima can be reduced to a Calculus I problem.

Summary

- We reviewed how to find local maxima and minima of functions of one variable
- We learned how to find local maxima, minima, and saddle points for functions of two variables using the gradient ∇f (to find critical points) and the determinant of the Hessian matrix (to classify the critical points as maxima, minima, or saddle points)
- We learned about the Extreme Value Theorem for a continuous function of two variables on a closed, bounded set