Math 213 - Double Integrals over General Regions

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Reminders

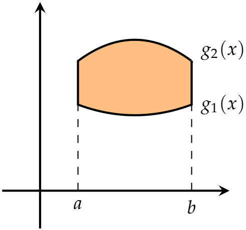
- Homework B7 on 14.8 (Lagrange multipliers) is due tonight!
- There is a drop-in review session for Exam II on Monday, October 14, 6:00-8:00 PM in KAS 213
- Exam II takes place on Wednesday, October 16, 5:00-7:00 PM

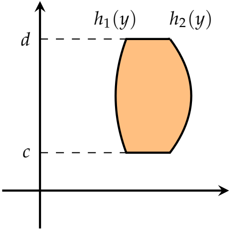
Unit II: Functions of Several Variables

Learning Goals

- Learn how to set up iterated integrals for double integrals over plane regions of Type I and Type II
- Learn properties of double integrals

Integrals Over General Regions





We'll see how to compute $\iint_R f(x,y)\,dA$ if R is one of the following kinds of regions:

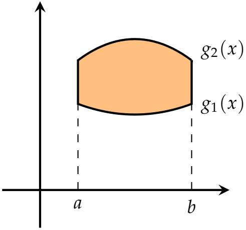
Type I: R lies between the graphs of two continuous functions of x

$$D = \{(x,y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

Type II: R lies between the graphs of two continuous functions of y

$$D = \{(x,y) : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$$

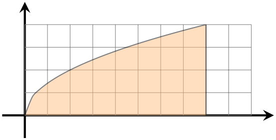
Double Integrals Over Type I Regions



To compute $\iint_D f(x,y)\,dA$ if

$$D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

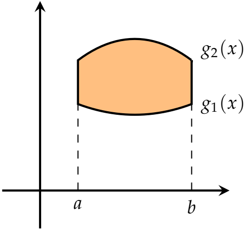
$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



Example: Find $\iint_D \frac{y}{x^2+1}\,dA$ if

$$D = \{(x,y) : 0 \le x \le 4, 0 \le y \le \sqrt{x}\}$$

Double Integrals Over Type I Regions



To compute $\iint_D f(x,y)\,dA$ if

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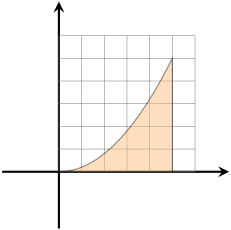
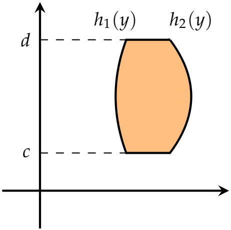


Example: Find $\iint_R x\cos y\,dA$ if D is the region bounded by $y = 0$, $y = x^2$, and $x = 1$

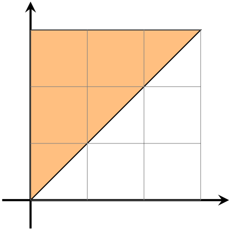
Double Integrals Over Type II Regions



To compute $\iint_D f(x,y)\,dA$ if

$$D = \{(x,y) : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$$

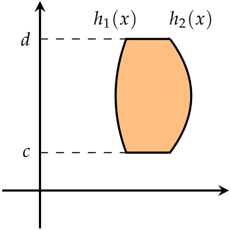
$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$



Example: Find $\iint_D e^{-y^2}\,dA$ if

$$D = \{(x,y) : 0 \leq y \leq 3,\ 0 \leq x \leq y\}$$

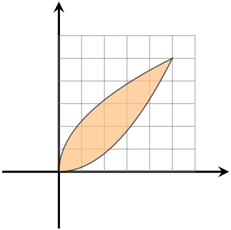
Double Integrals over Type II Regions



To compute $\iint_D f(x,y)\,dA$ if

$$D = \{(x,y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$



Example: Find the volume under the plane $3x + 3y - z = 0$ and above the region enclosed by the parabolas $y = x^2$ and $x = y^2$

Type I or Type II?

Find the best way to compute each of the following volumes.

1. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$
2. The volume enclosed by the cylinders $z = x^2$, $y = x^2$ and the planes $z = 0$, $y = 4$

Properties of Double Integrals, Part I

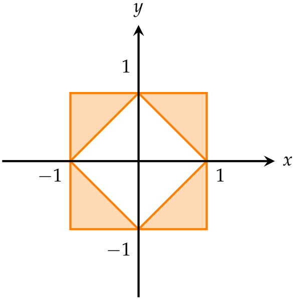
1. (linearity) $\iint_D [f(x,y)+g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$
2. (linearity) $\iint_D cf(x,y)\,dA = c\iint f(x,y)\,dA$
3. (order) If $f(x,y) \geq g(x,y)$ for all $(x,y) \in D$, then
$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$$
4. $\iint_D 1\,dA = A(D)$ where $A(D)$ is the area of the domain D

Find the volume of the the solid by subtracting two volumes:

The solid enclosed by the parabolic cylinders $y = 1 - x^2$, $y = x^2 - 1$ and the planes $x + y + z = 2$ and $2x + 2y - z + 10 = 0$

Properties of Double Integrals, II

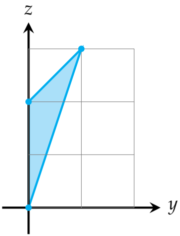
1. (*additivity) If $D = D_1 \cup D_2$, then
$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$
2. (order) If $m \leq f(x,y) \leq M$ then
$$mA(D) \leq \iint_D f(x,y)\,dA \leq MA(D)$$



Express $\iint_D xy\,dA$ as a union of type I and type II integrals if D is as shown

Volumes of Solids - Subtracting Two Volumes

We'll use the GeoGebra package at www.geogebra.org/3d to figure out what's going on here!



Find the volume of the solid enclosed by the parabolic cylinder

$$y = x^2$$

and the planes

$$z = 3y$$

and

$$z = 2 + y$$

Hint: It helps to consider the surface as two graphs $x = \pm\sqrt{y}$ over the yz plane!