Math 213 - Double Integrals in Polar Coordinates

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Reminders

- Homework B8 on Sections 15.1-15.2 is due tonight
- Happy Fall Break!

Unit III: Multiple Integrals, Vector Fields

Goals of the Day

- Review Polar Coordinates, introduce Polar Rectangles
- Learn how to compute double integrals over polar rectangles
- Learn how to compute double integrals over polar regions
- Learn to compute volumes using polar integrals

Reality Check

	Calculus I	**Calculus III**
Riemann sum	$\sum_{i=1}^{n} f(x_i^*)\Delta x$	$\sum_{i,j=1}^{n} f(x_i^*, y_j^*)\Delta A$
Riemann Integral	$\int_a^b f(x)\,dx$	$\iint_D f(x,y)\,dA$
Way of computing	$F(b) - F(a)$	Iterated Integral
Interpretation	Area under a curve	Volume under a surface

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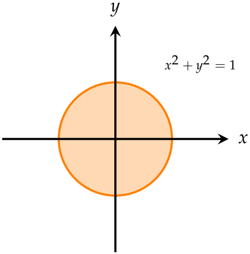
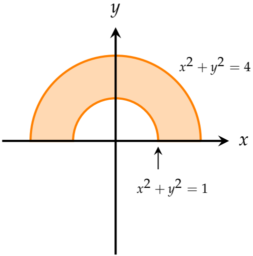
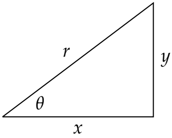
Review of Polar Coordinates







Recall that

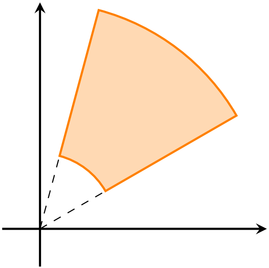
$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

and

$$x = r\cos\theta, \quad y = r\sin\theta.$$

How would you describe the regions at left in polar coordinates?

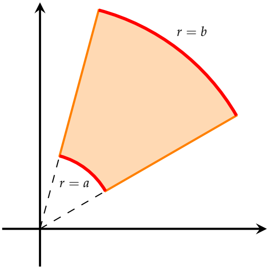
Polar Rectangles



A *polar rectangle* is a region

$$R = \{(r,\theta) : a \leq r \leq b, \quad \alpha \leq \theta \leq \beta\}.$$

Polar Rectangles



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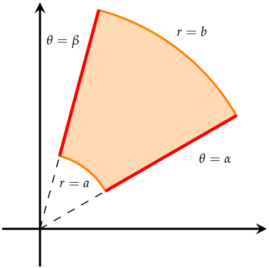

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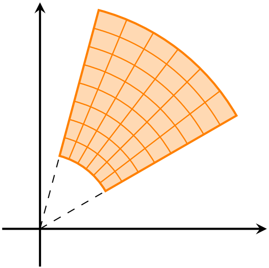
Polar Rectangles



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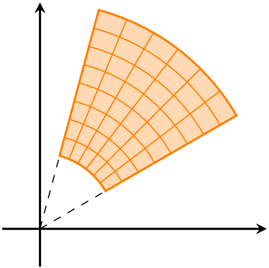
Like an ordinary rectangle a polar rectangle can be divided into *subrectangles*

Polar Rectangles



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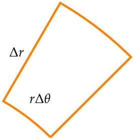
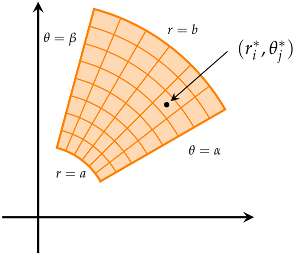
A small polar rectangle has area

$$\Delta A \simeq r \, \Delta r \, \Delta\theta$$



Integrals Over Polar Rectangles



The double integral $\iint_R f(x,y)\,dA$ is a limit of Riemann sums:

$$\sum_{i,j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, r_j \,\Delta r\,\Delta\theta$$

Rectangle R_{ij} is given by

$$R_{ij} = \{(r,\theta) : r_{i-1} \le r \le r_i, \theta_{j-1} \le \theta \le \theta_j\}$$

$$r_i = a + i\Delta r \quad , \theta_j = \alpha + j\Delta\theta$$

where

$$\Delta r = \frac{b-a}{n}, \quad \Delta\theta = \frac{\beta - \alpha}{n}$$

In the limit this leads to an iterated integral

$$\int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Integrals Over Polar Rectangles

Double Integral In Polar Coordinates The integral of a continuous function $f(x, y)$ over a polar rectangle R given by $a \leq r \leq b$, $\alpha \leq r \leq \beta$, is

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

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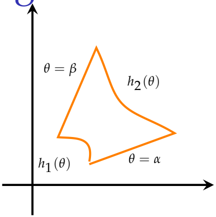
1. Find $\iint_R (2x - y)\,dA$ if R is the region in the first quadrant bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.
2. Find $\iint_R e^{-x^2-y^2}\,dA$ if D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis.

Integrals over Polar Regions



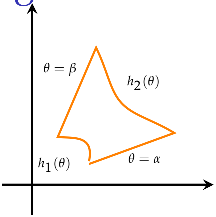
If f is continuous over a polar region of the form

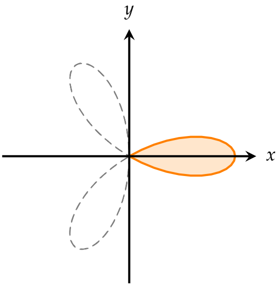
$$D = \{(r,\theta) : \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Integrals over Polar Regions





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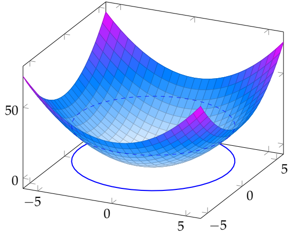
$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Find the area of one loop of the rose

$$r = \cos 3\theta$$

Volumes of Solids



Find the volume under the paraboloid

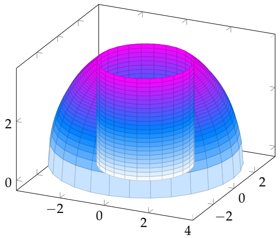
$$z = x^2 + y^2$$

and above the disc

$$x^2 + y^2 < 25$$

1. Describe the disc in polar coordinates
2. Transform $f(x, y)$ to polar coordinates

Volumes of Solids



Find the volume inside the sphere

$$x^2 + y^2 + z^2 = 16$$

and outside the cylinder

$$x^2 + y^2 = 4$$