Line Integrals over vector fields.

vector field. a continuous force field. (ex. grav'l field, electric force field).

$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ $\quad$ P, Q, R fns of (x, y, z)

$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$

$a \leq t \leq b.$

Recall work by a constant force $\vec{F}$ to move an object from point A to B is

$$W = \vec{F} \cdot \vec{D} \quad , \quad D = \vec{AB}.$$

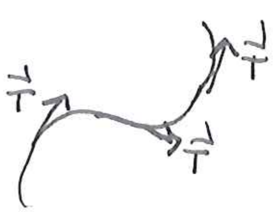
def. $\vec{F}$ continuous vector field defined on a smooth curve C: $\vec{r}(t)$, $a \leq t \leq b$.

The line integral over the vector field $\vec{F}$ on curve C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \vec{F} \cdot \vec{T}\, ds$$

or unit tangent vector



$\vec{T} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}.$

Note cancellation.

$\left(= \int_a^b \vec{F} \cdot \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \cdot |\vec{r}\,'(t)|\, dt \right)$

line integral over vect. field. 2.

ex $\displaystyle\int_C \vec{F} \cdot d\vec{r}$

$$\vec{F} = 8x^2yz\,\vec{i} + 5z\,\vec{j} - 4xy\,\vec{k}$$

$$C: \quad \vec{r}(t) = \underbrace{t\,\vec{i} + t^2\,\vec{j} + t^3\,\vec{k}}_{\text{the moment curve.}} \quad , 0 \le t \le 1$$

$$\vec{F}(\vec{r}(t)) = 8t^2(t^2)(t^3)\,\vec{i} + 5t^3\,\vec{j} - 4t(t^2)\,\vec{k}$$

$$= \langle 8t^7, \; 5t^3, -4t^3 \rangle .$$

$$\vec{r}\,'(t) = \vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 8t^7 + 10t^4 - 12t^5$$

So

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 8t^7 + 10t^4 - 12t^5 \, dt$$

$$= t^8 + 2t^5 - 2t^6 \Big]_0^1$$

$$= 1 + 2 - 2 = 1 .$$

line integral
over vector field.

line integral
wrt x, y + z.

Lemma: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$

where $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$.

$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$

Proof

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b (P\vec{i} + Q\vec{j} + R\vec{k})\cdot(x'(t)\,\vec{i} + y'(t)\,\vec{j} + z'(t)\,\vec{k})\,dt.$$

$$= \int_a^b Px' + Qy' + Rz' \; dt.$$

$$= \int_a^b Px'\,dt + \int_a^b Qy'\,dt + \int_a^b Rz'\,dt$$

$$= \int_C P\,dx + Q\,dy + R\,dz. \quad \boxed{q}$$