Math 213 - Fundamental Theorem for Line Integrals

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November 15, 2019

Reminders

- Homework C7 is due tonight
- Thanksgiving is coming!

Unit IV: Vector Calculus

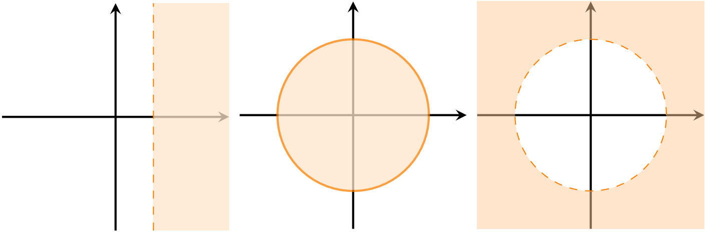
Goals of the Day

- Learn the Vocabulary for Section 16.3
- Learn the Fundamental Theorem for Line Integrals
- Learn what it means for a line integral to be *independent of path*
- Learn how to tell when a vector field $\mathbf{F}$ is *conservative* and how to find the function f with $\nabla f = \mathbf{F}$

Vocabulary - Open Regions

open region A region D of $\mathbb{R}^2$ or $\mathbb{R}^3$ where for every point P in the region, there is a disc or sphere centered at P contained in D

Which of the following regions is open?



$\{(x,y) : x > 1\}$ $\{(x,y) : x^2 + y^2 \leq 4\}$ $\{(x,y) : x^2 + y^2 > 4\}$

Chain Rule Puzzler

If $f(x,y,z)$ is a function and $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a parameterized curve, what is

$$\frac{d}{dt}\left[f(x(t), y(z), z(t))\right]$$

in terms of ∇f and $\mathbf{r}'(t)$?

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in terms of ∇f and $\mathbf{r}'(t)$?

Answer: $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$

Remember the Fundamental Theorem of Calculus?

What is

$$\int_a^b \frac{d}{dt} F(t)\, dt \ ?$$

(Remember the *Net Change Theorem*?)

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(Remember the *Net Change Theorem*?)

Answer: $F(b) - F(a)$

Line Integral of a Gradient Vector Field

Suppose $\mathbf{F} = \nabla f$ for a potential function $f(x, y, z)$

Suppose $\mathbf{r}(t)$, $a \leq t \leq b$ is a parameterized path C.

Is there a simple way to compute

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_a^b \frac{d}{dt}\left(f(\mathbf{r}(t))\right)\, dt$$

like the one-variable “net change theorem”?

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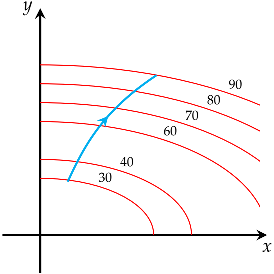
Answer: You bet!

Line Integral of a Gradient Vector Field

Theorem Suppose that $\mathbf{F}(\mathbf{r}) = \nabla f(\mathbf{r})$ is a gradient vector field, and C is a path parameterized by $\mathbf{r}(t)$, $a \leq t \leq b$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

How to think about the Fundamental Theorem for Line Integrals



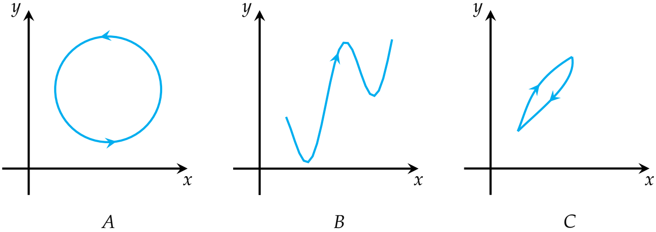
The figure at the left shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.

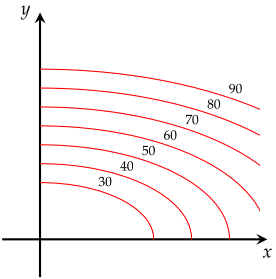
Hint: Think of f as a height function, and the contour plot as a contour map. The gradient gives the magnitude and direction of the greatest change in height at any given point.

Vocabulary - Paths and Vector Fields

path	A piecewise smooth curve
closed path	A curve whose initial and terminal points are the same
conservative vector field	A vector field $\mathbf{F}$ which is the gradient of a scalar function f, called the *potential*, so that $\mathbf{F} = \nabla f$

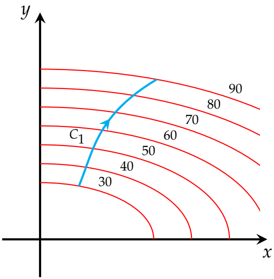
Which of the following is not a closed path?





At left is the contour plot for a function f whose gradient is continuous.

Compute the following:



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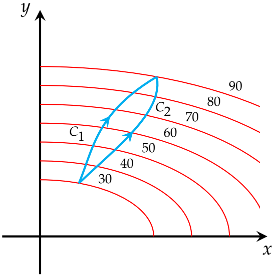
Compute the following:

- $\int_{C_1} \nabla f \cdot d\mathbf{r}$



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- $\int_{C_1} \nabla f \cdot d\mathbf{r}$
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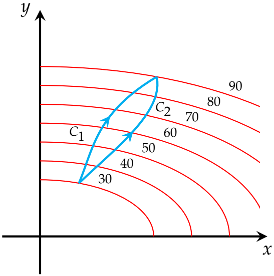

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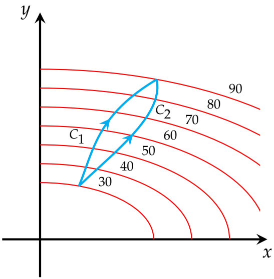
Compute the following:

- $\int_{C_1} \nabla f \cdot d\mathbf{r}$
- $\int_{C_2} \nabla f \cdot d\mathbf{r}$
- Does it matter what path connects the endpoints?



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Compute the following:

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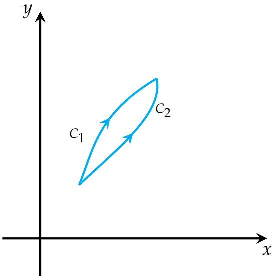
Definition A line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is *independent of path* in a domain D f

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

for any two paths C_1 and C_2 that have the same initial and terminal points.

Path Independence and Closed Paths

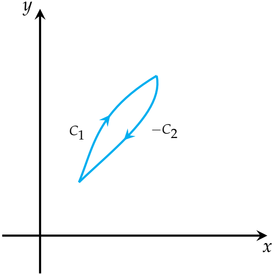


If

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

and we reverse the direction of C_2 . . .

Path Independence and Closed Paths



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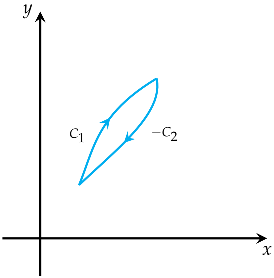
Then
$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$
where C is the closed loop path that starts with C_1 and ends with $-C_2$.

Path Independence and Closed Paths



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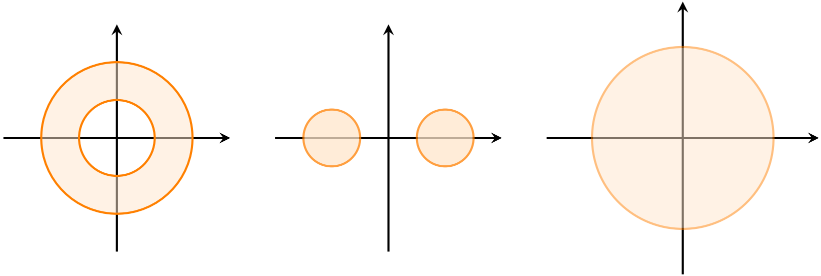
Theorem The integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path for all paths in a domain D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path in D.

Vocabulary - Connected Regions

connected region A region D of $\mathbb{R}^2$ or $\mathbb{R}^3$ where any points P and Q can be connected by a path contained in D

domain An open, connected region of $\mathbb{R}^2$ or $\mathbb{R}^3$

Which of these regions is *not* connected?



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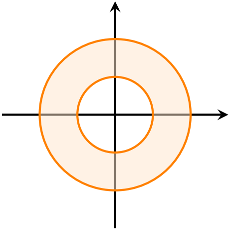
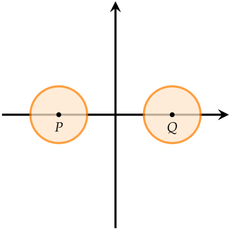
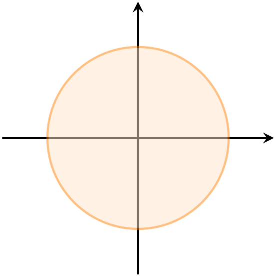




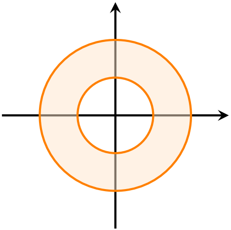


Vocabulary - Simply Connected Regions

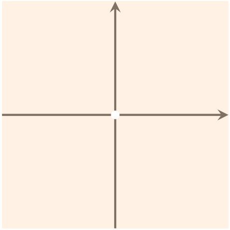
simple curve A curve that doesn't intersect itself

simply connected A connected region so that every simple closed curve in D surrounds only points of D

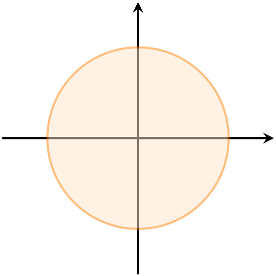
Which of these regions is *not* simply connected?



$\{(x,y) : 1 \leq x^2 + y^2 \leq 2\}$



$\{(x,y) : (x,y) \neq (0,0)\}$



$\{(x,y) : x^2 + y^2 \leq 4\}$

First Theorem of the Day

Theorem Suppose $\mathbf{F}$ is a vector field that is continuous on an open, simply connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there is a function f so that $\nabla f = \mathbf{F}$

How do you find the function f (two dimensions)?

- Pick a point (a, b) in the domain D
- Compute

$$f(x, y) = \int_{(a,b)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}$$

- In fact, you can show that this function f satisfies

$$\mathbf{F}(x, y) = \frac{\partial f}{\partial x}(x, y)\mathbf{i} + \frac{\partial f}{\partial y}(x, y)\mathbf{j}$$

How You (Almost) Tell when F is Conservative

Key Observation If $F = \nabla f$ then

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} = \frac{\partial f}{\partial x}(x,y)\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Compute $\partial P/\partial y$ and $\partial Q/\partial x$ as a second derivative of f:

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Compute $\partial P/\partial y$ and $\partial Q/\partial x$ as a second derivative of f:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$$

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$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$$

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So, by Clairaut's Theorem, for a conservative vector field:

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

Find the Conservative Vector Field

Theorem If $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a conservative vector field, and P, Q have continuous first-order partials on a domain D, then throughout D

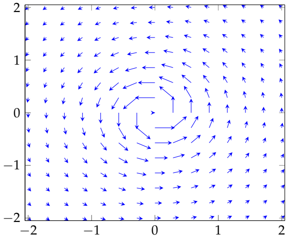
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Which of the following vector fields are definitely *not* conservative?

1. $\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$
2. $\mathbf{F}(x,y) = x^3\mathbf{i} + y^2\mathbf{j}$
3. $\mathbf{F}(x,y) = ye^x\mathbf{i} + (e^x + e^y)\mathbf{j}$
4. $\mathbf{F}(x,y) = \dfrac{-y}{x^2+y^2}\mathbf{i} + \dfrac{x}{x^2+y^2}\mathbf{j}, \quad (x,y) \neq (0,0)$

There's One in Every Crowd

$$\mathbf{F}(x,y) = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$$



1. Does $\mathbf{F}$ satisfy the "conservative vector field" condition?
2. Suppose C is the circle $x^2 + y^2 = 1$. What is $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field shown?
3. Is the domain

$$\{(x,y) : x^2 + y^2 \neq 0\}$$

simply connected?

Second Theorem of the Day

Theorem Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field defined on an open, simply connected region D. Suppose that P and Q have continuous partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout D. Then $\mathbf{F}$ is conservative.

Which of the following vector fields are conservative?

1. $\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$
2. $\mathbf{F}(x,y) = x^3\mathbf{i} + y^2\mathbf{j}$
3. $\mathbf{F}(x,y) = ye^x\mathbf{i} + (e^x + e^y)\mathbf{j}$
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How to Find the Potential f

Recall that if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} = \nabla f$, then

$$P = \frac{\partial f}{\partial x}, \quad Q = \frac{\partial f}{\partial y}$$

Example Find f if $\mathbf{F}(x, y) = (y^2 - 2x)\mathbf{i} + 2xy\mathbf{j}$

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1. $\dfrac{\partial f}{\partial x} = y^2 - 2x$ so taking antiderivatives in x

$$f(x,y) = y^2 x - x^2 + C(y)$$

where $C(y)$ is a constant *that may depend on y*

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2. From the answer we found in step 1, $\dfrac{\partial f}{\partial y} = 2xy + C'(y) = 2xy$ so $C'(y) = 0$
3. Finally, $f(x,y) = xy^2 - x^2 + C$

Line Integrals of Conservative Vector Fields

Recall that if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} = \nabla f$, then

$$P = \frac{\partial f}{\partial x}, \quad Q = \frac{\partial f}{\partial y}$$

Example: Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ by finding f so that $\nabla f = \mathbf{F}$ if:

$$\mathbf{F}(x, y) = (1 + xy)e^{xy}\mathbf{i} + x^2 e^{xy}\mathbf{j}$$
$$C : \mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad 0 \leq t \leq \pi/2$$