Math 213 - Green's Theorem

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November 18, 2019

Reminders

- Quiz #9 on 16.1-16.2 (vector fields, line integrals) takes place on Thursday
- Homework D1 is due on Friday of this week
- Thanksgiving is coming!

Unit IV: Vector Calculus

Goals of the Day

- Understand positive and negative orientations of a simple, closed, plane curve
- Understand Green's Theorem
- Use Green's Theorem to compute line integrals and area integrals
- Understand how Green's Theorem connects with Conservative Vector Fields

Who Was Green?

George Green (1793–1841) was a British mathematical physicist who studied electricity and magnetism. From Wikipedia (where else?):

> *Green was the first person to create a mathematical theory of electricity and magnetism and his theory formed the foundation for the work of other scientists such as James Clerk Maxwell, William Thomson, and others. His work on potential theory ran parallel to that of Carl Friedrich Gauss.*
>
> *Green's life story is remarkable in that he was almost entirely self-taught. He received only about one year of formal schooling as a child, between the ages of 8 and 9.*

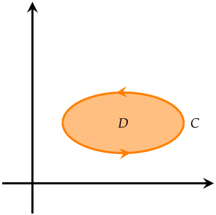
This last word on Green comes from the Mactutor History of Mathematics' article about him:

> *Through Thomson [Lord Kelvin], [James Clerk] Maxwell, and others, the general mathematical theory of potential developed by an obscure, self-taught miller's son would lead to the mathematical theories of electricity underlying twentieth-century industry.*

Closed Curves, Simple Curves, Oriented Curves

region	An open subset of the xy plane
closed curve	A curve whose initial and terminal points are the same
simple curve	A curve that doesn't intersect itself
simply connected region	A connected region so that every simple closed curve in D surrounds only points of D

A simple closed curve C surrounding a region D is *positively oriented* (or has *positive orientation*) if the curve C traverses D *counterclockwise* with the enclosed region to the *left*

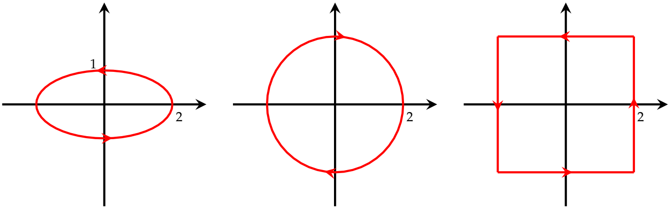


At left, the region D is surrounded by the positively oriented curve C. Which of these parameterizations gives C the correct orientation?

1. $x(t) = 1 + 1.5\cos(t), \quad y(t) = 0.5\sin t,$ $0 \leq t \leq 2\pi$
2. $x(t) = 1 + 1.5\cos(2\pi - t)$ $y(t) = 0.5\sin(2\pi - t),$ $0 \leq t \leq 2\pi$

Orientation Implies Parameterization

In what we do this week, it will be important to *parameterize* curves so that you get the *orientation* right. Can you give a correct parameterization for each of the following oriented curves? Which is positively oriented, and which is negatively oriented?



Green's Theorem

Green's Theorem Let C be a positive oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives in an open region that contains D, then

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy$$

Compare this to the Fundamental Theorem of Calculus, Part 2 from Math 113:

If F is continuous on $[a,b]$ and differentiable in (a,b), then

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Green's Theorem

Compare the formulas:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy \qquad \text{(Green's Theorem)}$$

$$\int_a^b F'(x)\,dx = F(b) - F(a) \qquad \text{(FTC, Part II)}$$

- In Green's theorem, C bounds the region D, sometimes written $C = \partial D$
- In FTC, the endpoints a and b bound the interval $[a, b]$

- In Green's theorem, the integral of a 'derivative' of the vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ equals the line integral of the vector field $\mathbf{F}$ over the boundary
- In FTC, the integral of the derivative of F equals a difference of values of F over the boundary

Green's Theorem

The integral around a *closed* curve C of $P(x,y)\,dx + Q(x,y)\,dy$ is sometimes denoted

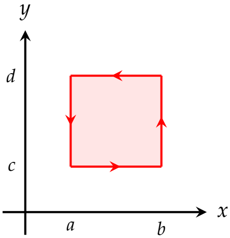
$$\oint_C P(x,y)\,dx + Q(x,y)\,dy$$

With this notation, the main formula in Green's theorem says

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C P\,dx + Q\,dy$$

where C is a positively oriented curve that bounds D

Green's Theorem in a Special Case

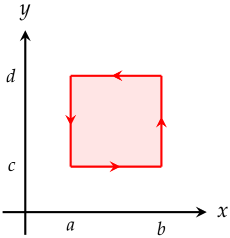


Suppose D is the rectangle $[a, b] \times [c, d]$.

The contour C consists of the four line segments shown.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Green's Theorem in a Special Case

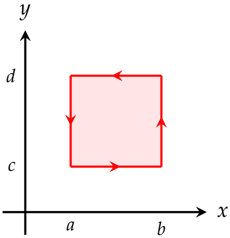


Suppose D is the rectangle $[a, b] \times [c, d]$.

The contour C consists of the four line segments shown.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$
$$= \int_c^d \left(\int_a^b \frac{\partial Q}{\partial x} \, dx \right) dy - \int_a^b \left(\int_c^d \frac{\partial P}{\partial y} \, dy \right) dx$$

Green's Theorem in a Special Case



Suppose D is the rectangle $[a,b] \times [c,d]$.

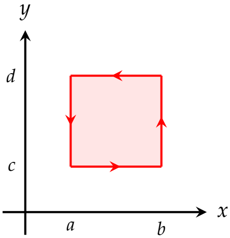
The contour C consists of the four line segments shown.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \int_c^d \left(\int_a^b \frac{\partial Q}{\partial x}\, dx \right) dy - \int_a^b \left(\int_c^d \frac{\partial P}{\partial y}\, dy \right) dx$$

$$= \int_c^d \left(Q(b,y) - Q(a,y) \right) dy - \int_a^b \left(P(x,d) - P(x,c) \right) dx$$

Green's Theorem in a Special Case



Suppose D is the rectangle $[a, b] \times [c, d]$.

The contour C consists of the four line segments shown.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \int_c^d \left(\int_a^b \frac{\partial Q}{\partial x}\, dx \right) dy - \int_a^b \left(\int_c^d \frac{\partial P}{\partial y}\, dy \right) dx$$

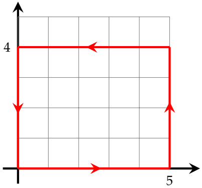
$$= \int_c^d \left(Q(b, y) - Q(a, y) \right) dy - \int_a^b \left(P(x, d) - P(x, c) \right) dx$$

$$= \int_a^b P(x, c)\, dx + \int_c^d Q(b, y)\, dy - \int_a^b P(x, d)\, dx - \int_c^d Q(a, y)\, dy$$

Using Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C P\,dx + Q\,dy$$

where C is a positively oriented curve that bounds D



Evaluate directly and then use Green's theorem to find

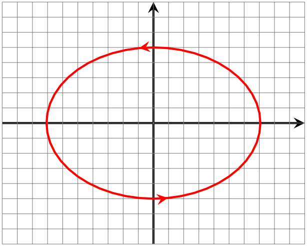
$$\oint_C y^2\,dx + x^2 y\,dy$$

if C is the path shown at left

Using Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C P\,dx + Q\,dy$$

where C is a positively oriented curve that bounds D



Use Green's theorem to find

$$\oint_C y^4\,dx + 2xy^3\,dy$$

if C is the ellipse $x^2 + 2y^2 = 2$

Using Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C P\,dx + Q\,dy$$

where C is a positively oriented curve that bounds D

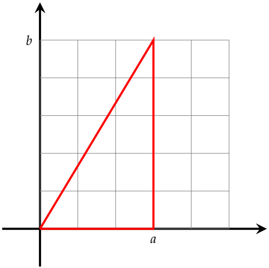
Recall that if D has area A,

$$\begin{aligned} \overline{x} &= \frac{1}{A} \int_D x\,dA \\ \overline{y} &= \frac{1}{A} \int_D y\,dA \end{aligned}$$

Using Green's theorem, show that

$$\begin{aligned} \overline{x} &= \frac{1}{2A} \oint_C x^2\,dy \\ \overline{y} &= -\frac{1}{2A} \oint_C y^2\,dx \end{aligned}$$

Using Green's Theorem



Using the formulas

$$\overline{x} = \frac{1}{2A} \oint_C x^2\,dy$$
$$\overline{y} = -\frac{1}{2A} \oint_C y^2\,dx$$

find the centroid of the triangle shown at left.

Green's Theorem and Conservative Vector Fields

We can now prove a Theorem from a previous lecture about conservative vector fields.

Theorem Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open, simply connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ throughout } D.$$

Then $\mathbf{F}$ is conservative.

Proof. If C is any closed path, and D is the domain it encloses,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0.$$

so the integral around *any* closed path in D is zero. This means we can pick a point (a, b) in D and define

$$f(x, y) = \int_{(a,b)}^{(x,y)} P\,dx + Q\,dy.$$

The function $f(x, y)$ satisfies $\nabla f(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$.