Math 213 - Curl and Divergence

Peter A. Perry

University of Kentucky

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Reminders

- Quiz #9 on 16.1-16.2 (vector fields, line integrals) takes place tomorrow
- Homework D1 is due on Friday of this week
- Thanksgiving is coming!

Unit IV: Vector Calculus

Fundamental Theorem for Line Integrals

Green's Theorem

Curl and Divergence

Parametric Surfaces and their Areas

Surface Integrals

Stokes' Theorem, I

Stokes' Theorem, II

The Divergence Theorem

Review

Review

Review

Goals of the Day

This lecture is about two very important 'derivatives' of a vector field. You'll learn:

- How to compute the *curl* of a vector field and what it measures
- How to compute the *divergence* of a vector field and what it measures
- (Sneak preview) The theorems that give the meaning of divergence and curl

Curl

If $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$, and the partial derivatives of P, Q, and R all exist, then the *curl* of $\mathbf{F}$ is a new vector field:

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

This new vector field measures the "rotation" of the vector field $\mathbf{F}$ at a given point (x, y, z):

- Its *direction* is the axis of rotation, dictated by the right-hand rule
- Its *magnitude* is the angular speed of rotation

Curl

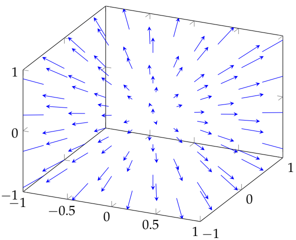
$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

The new vector field $\operatorname{curl} \mathbf{F}$ is sometimes written $\nabla \times \mathbf{F}$ because of an easier-to-remember formula:

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Curl

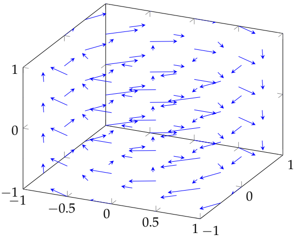
$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$



If $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}$$

$\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + 0\mathbf{k}$



If $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + 0\mathbf{k}$ then

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\mathbf{k}$$

Curl

A gradient vector field has *zero curl*:

$$\nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \mathbf{i} + \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \mathbf{j} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \mathbf{k}$$

$$= \mathbf{0}$$

so the curl “detects” conservative vector fields.

Divergence

The *divergence* of a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a *scalar* function:

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

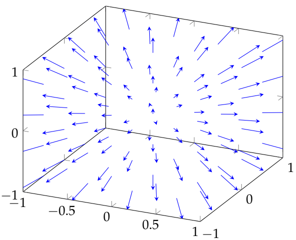
Sometimes $\operatorname{div} \mathbf{F}$ is written $\nabla \cdot \mathbf{F}$:

$$\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The divergence computes the outflow per unit volume of the vector field (thought of as a velocity field)

Divergence

$\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

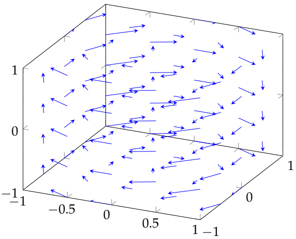


If $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then

$$\nabla \cdot \mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

(same outflow at each point of space)

$\mathbf{F}(x,y,z) = y\mathbf{i} - x\mathbf{j} + 0\mathbf{k}$



If $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + 0\mathbf{k}$ then

$$\nabla \cdot \mathbf{F} = \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} + 0 = 0$$

(no outflow anywhere in space)

Divergence

Remember that $\nabla \times (\nabla f) = 0$? There is an analogous result for the divergence:

$$\operatorname{div}\operatorname{curl}\mathbf{F} = 0$$

You can see this using the definitions of divergence and curl:

$$\operatorname{div}\operatorname{curl}\mathbf{F} = \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

The second partial derivatives cancel in pairs by Clairaut's theorem.

It turns out that *any vector field* **F** can be written as

$$\mathbf{F} = \nabla f + \nabla \times \mathbf{A}$$

for a *scalar potential* f and a *vector potential* **A**.

Divergence and Curl

If f is a scalar function and $\mathbf{F}$ is a vector function, which of these expressions make sense? Do they define a scalar or a vector? Remember that

- curl $\mathbf{F}$ is a vector
- div $\mathbf{F}$ is a scalar

(a)	curl f	(b)	grad f
(c)	div $\mathbf{F}$	(d)	curl(grad f)
(e)	grad $\mathbf{F}$	(f)	grad(div $\mathbf{F}$)
(g)	div(grad f)	(h)	grad(div f)
(i)	curl(curl $\mathbf{F}$)	(j)	div(div f)

Conservative Vector Fields Again

Determine whether the vector field

$$\mathbf{F}(x, y, z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$$

is conservative and, if so, find a function f so that $\nabla f = \mathbf{F}$.

Vector Identities

Show that $\text{div}(f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f \, \text{div} \, \mathbf{F}$ where $\mathbf{F} = \langle P, Q, R \rangle$

What is the right 'product rule' for $\text{curl}(f\mathbf{F})$?

What is $\text{div} \, \text{grad} \, f$?

Vector Identities

Show that $\operatorname{div}(f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f \operatorname{div} \mathbf{F}$ where $\mathbf{F} = \langle P, Q, R \rangle$

$$\begin{aligned}\operatorname{div}(f\mathbf{F}) &= \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR) \\ &= \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R + f\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \\ &= \operatorname{grad} f \cdot \mathbf{F} + f \operatorname{div} \mathbf{F}\end{aligned}$$

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What is $\operatorname{div} \operatorname{grad} f$?

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What is the right 'product rule' for $\text{curl}(f\mathbf{F})$?

$$\text{curl}(f\mathbf{F}) = \text{grad}\, f \times \mathbf{F} + f \, \text{curl}(\mathbf{F})$$

What is $\text{div}\, \text{grad}\, f$?

Vector Identities

Show that $\text{div}(f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f \,\text{div}\,\mathbf{F}$ where $\mathbf{F} = \langle P, Q, R \rangle$

$$\begin{aligned}
\text{div}(f\mathbf{F}) &= \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR) \\
&= \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R + f\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \\
&= \text{grad}\, f \cdot \mathbf{F} + f\,\text{div}\,\mathbf{F}
\end{aligned}$$

What is the right ‘product rule’ for $\text{curl}(f\mathbf{F})$?

$$\text{curl}(f\mathbf{F}) = \text{grad}\, f \times \mathbf{F} + f\,\text{curl}(\mathbf{F})$$

What is $\text{div}\,\text{grad}\, f$?

$$\begin{aligned}
\text{div}\,\text{grad}\, f &= \text{div}\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \\
&= \frac{\partial}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial}{\partial y}\frac{\partial f}{\partial y} + \frac{\partial}{\partial z}\frac{\partial f}{\partial z} \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

Divergence Theorem, Stokes' Theorem

Divergence Theorem Suppose E is a simple solid region and S is its boundary. Let $\mathbf{N}$ be the outward normal to S. Then

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

Stokes' Theorem Suppose S is an oriented piecewise-smooth surface with outward normal $\mathbf{N}$, bounded by a simple closed curve C with piecewise smooth boundary. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{N}\, dS$$

The 'Big Three' Theorems of Vector Calculus

$$F(b) - F(a) = \int_a^b F'(x)\,dx \qquad \text{(Fundamental)}$$

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \qquad \text{(Green)}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{N}\,dS \qquad \text{(Stokes)}$$

$$\iint_S \mathbf{F} \cdot \mathbf{N}\,dS = \iiint_E \operatorname{div} \mathbf{F}\,dV \qquad \text{(Divergence)}$$