Math 213 - Surface Integrals

Peter A. Perry

University of Kentucky

November 25, 2019

Reminders

- Homework D2 is due **tonight**
- Thanksgiving is coming!

Unit IV: Vector Calculus

Goals of the Day

This lecture is about parametric surfaces. You'll learn:

- How to integrate a scalar function over a parameterized surface
- What an *oriented surface* is and how to compute its *unit normal*
- How to integrate a vector field over a parameterized surface

Sneak Preview - Scalar Surface Integrals

Scalar Line Integrals

If C is parameterized by

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle,$$

$a \leq t \leq b$:

$$\int_C F\, ds = \int_a^b F(x(t), y(t), z(t)) ds$$

where

$$ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

Scalar Surface Integrals

If S is parameterized by

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle,$$

$(u, v) \in D$:

$$\iint_S F\, dS = \iint_D F |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$$

where

$$F = F(x(u, v), y(u, v), z(u, v))$$

$$\mathbf{r}_u = \frac{\partial}{\partial u} \mathbf{r}(u, v)$$

$$\mathbf{r}_v = \frac{\partial}{\partial v} \mathbf{r}(u, v)$$

Sneak Preview - Vector Surface Integrals

Vector Line Integrals

If C is parameterized by

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle,$$

$a \leq t \leq b$:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{T}(t)\, ds$$

$$\mathbf{F} = \mathbf{F}(x(t), y(t), z(t))$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Vector Surface Integrals

If S is parameterized by

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle,$$

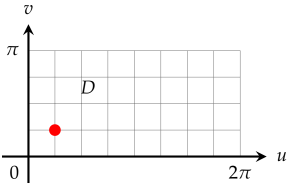
$(u,v) \in D$:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot \mathbf{N}\, du\, dv$$

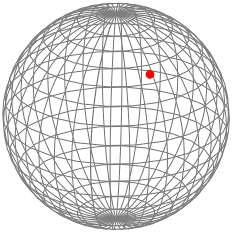
$$\mathbf{F} = \mathbf{F}(x(u,v), y(u,v), z(u,v))$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$$

Parameterized Surface Review



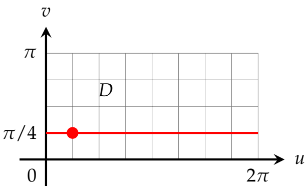
The parameter space D



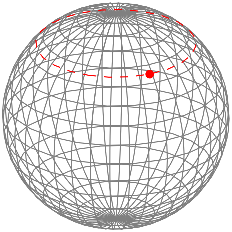
$\sin v \cos u \mathbf{i} + \sin v \sin u \mathbf{j} + \cos v \mathbf{k}$

A *parameterized surface* is traced out by $\mathbf{r}(u, v)$ where $(u, v) \in D$, a region in the uv plane.

Parameterized Surface Review



The parameter space D

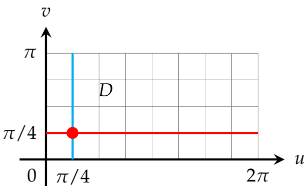


$\sin v \cos u\mathbf{i} + \sin v \sin u\mathbf{j} + \cos v\mathbf{k}$

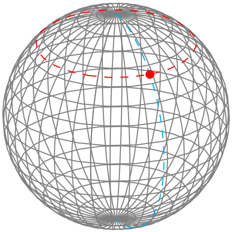
A *parameterized surface* is traced out by $\mathbf{r}(u, v)$ where $(u, v) \in D$, a region in the uv plane.

If v is held fixed and u varies, the result is a curve along the surface.

Parameterized Surface Review



The parameter space D



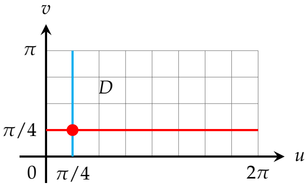
$\sin v \cos u\mathbf{i} + \sin v \sin u\mathbf{j} + \cos v\mathbf{k}$

A *parameterized surface* is traced out by $\mathbf{r}(u, v)$ where $(u, v) \in D$, a region in the uv plane.

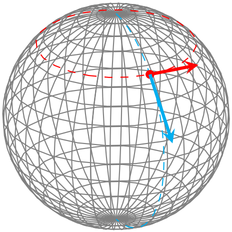
If v is held fixed and u varies, the result is a curve along the surface.

If u is held fixed and v varies, the result is a different curve along the surface.

Parameterized Surface Review



The parameter space D


$\sin v \cos u\mathbf{i} + \sin v \sin u\mathbf{j} + \cos v\mathbf{k}$

A *parameterized surface* is traced out by $\mathbf{r}(u, v)$ where $(u, v) \in D$, a region in the uv plane.

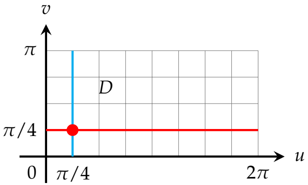
If v is held fixed and u varies, the result is a curve along the surface.

If u is held fixed and v varies, the result is a different curve along the surface.

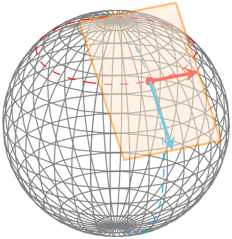
Each curve has a tangent vector, so there are two independent tangent vectors

$$\mathbf{r}_u = \partial\mathbf{r}/\partial u, \quad \mathbf{r}_v = \partial\mathbf{r}/\partial v$$

Parameterized Surface Review



The parameter space D



$\sin v \cos u \mathbf{i} + \sin v \sin u \mathbf{j} + \cos v \mathbf{k}$

A *parameterized surface* is traced out by $\mathbf{r}(u, v)$ where $(u, v) \in D$, a region in the uv plane.

If v is held fixed and u varies, the result is a curve along the surface.

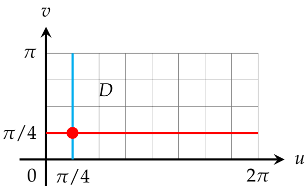
If u is held fixed and v varies, the result is a different curve along the surface.

Each curve has a tangent vector, so there are two independent tangent vectors

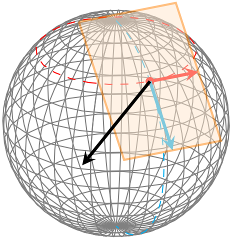
$$\mathbf{r}_u = \partial \mathbf{r}/\partial u, \quad \mathbf{r}_v = \partial \mathbf{r}/\partial v$$

The vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ span a tangent plane

Parameterized Surface Review



The parameter space D



$\sin v \cos u\mathbf{i} + \sin v \sin u\mathbf{j} + \cos v\mathbf{k}$

A *parameterized surface* is traced out by $\mathbf{r}(u, v)$ where $(u, v) \in D$, a region in the uv plane.

If v is held fixed and u varies, the result is a curve along the surface.

If u is held fixed and v varies, the result is a different curve along the surface.

Each curve has a tangent vector, so there are two independent tangent vectors

$$\mathbf{r}_u = \partial\mathbf{r}/\partial u, \quad \mathbf{r}_v = \partial\mathbf{r}/\partial v$$

The vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ span a tangent plane

The normal to the tangent plane is

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$$

Scalar Surface Integrals

If S is parameterized by $\mathbf{r}(u,v)$ for $(u,v) \in D$, and f is a function continuous in a neighborhood of S,

$$\iint_S f(x,y,z)\,dS = \iint_D F(x(u,v),y(u,v),z(u,v))\ |\mathbf{r}_u \times \mathbf{r}_v|\ du\,dv$$

1. Suppose that $f(x,y,z) = g(\sqrt{x^2+y^2+z^2})$ where $g(2) = 5$. Find $\iint_S f(x,y,z)\,dS$ if S is the sphere $x^2+y^2+z^2 = 4$.
2. Find $\iint_S xz\,dS$ if S is the part of the plane $2x+2y+z=4$ that lies in the first octant.
3. Find $\iint y^2\,dS$ if S is the part of the sphere $x^2+y^2+z^2=1$ that lies above the cone $z = \sqrt{x^2+y^2}$.

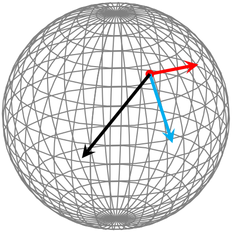
The Oriented Unit Normal to a Surface

A surface S is called an *oriented surface* if there is a unit normal vector $\mathbf{n}$ at every point on the surface that varies continuously along the surface. Every parameterized surface has such a unit normal, given by

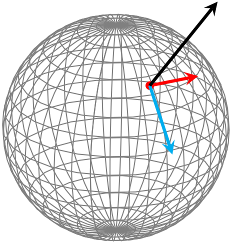
$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

Every orientable surface in $\mathbb{R}^3$ has *two* possible orientations, one with $\mathbf{n}$ and the other with $-\mathbf{n}$.

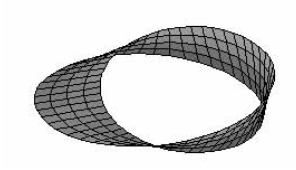
The orientation of the sphere with $\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$



The orientation of the sphere with $\mathbf{n} = -\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$



Oriented Surfaces versus the Möbius Strip



The Möbius Band

August Ferdinand Möbius (1790–1868)

Vector Surface Integrals

If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, the *surface integral of* $\mathbf{F}$ *over* S is

$$\iint_S \mathbf{F}\, d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

This integral is also called the *flux* of $\mathbf{F}$ across S. Depending on the choice of normal, it measures either what goes *in* (inward normal) or what comes *out* (outward normal).

1. Find the flux of $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$ across S if S is the sphere of radius 1 and center at the origin
2. Find the flux of $\mathbf{F}(x, y, z) = y\mathbf{j} - z\mathbf{k}$ across the paraboloid $y = x^2 + z^2$, $0 \leq y \leq 1$, and the disc $x^2 + y^2 \leq 1$, $y = 1$