Math 213 - The Gauss Divergence Theorem

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Reminders

- Homework D4 (Surface integrals, Stokes' Theorem) is due tonight

Unit IV: Vector Calculus

Fundamental Theorem for Line Integrals

Green's Theorem

Curl and Divergence

Parametric Surfaces and their Areas

Surface Integrals

Stokes' Theorem, I

Stokes' Theorem, II

The Divergence Theorem

Review

Review

Review

Goals of the Day

This lecture is about the Gauss Divergence Theorem, which illuminates the meaning of the divergence of a vector field. You will learn:

- How the flux of a vector field over a surface bounding a *simple volume* to the divergence of the vector field in the enclosed volume
- How to compute the flux of a vector field by integrating its divergence

Vector (Differential) Calculus: The Story So Far

We have defined two 'derivatives' of a vector field **F**. One is a scalar and the other is a vector.

The *divergence* of a vector field

$$\mathbf{F} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

is the scalar

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The *curl* of a vector field **F** is

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Vector (Integral) Calculus: The Story So Far

The *circulation* of a vector field **F** around a closed curve C is the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

Stokes' Theorem relates the circulation of a vector field **F** over a curve C to the surface integral of its curl over any surface that bounds C:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

The *flux* of a vector field through a surface S bounding a volume $E$is the surface integral

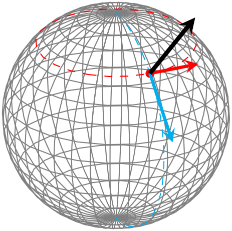
$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

where **n** is the *outward* normal. The *divergence theorem*, which we'll study today, relates the flux of **F** to the integral of its divergence.

What's A *Simple Volume*?

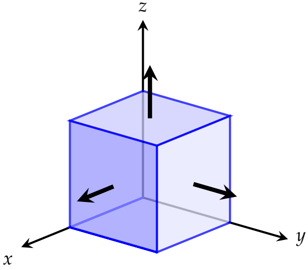
If volume E is a simple volume if it has no holes and its boundary separates $\mathbb{R}^3$ into an "inside" and an "outside."

The sphere of radius a centered at $(0, 0, 0)$



$\sin v \cos u\mathbf{i} + \sin v \sin u\mathbf{j} + \cos v\mathbf{k}$

The box of side a in the first octant



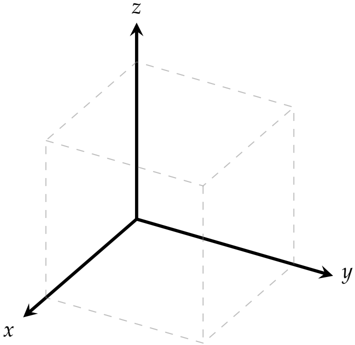
The Flux of a Vector Field out of a Box

What is the flux of a vector field

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

out of a box of side a?

There are *six* surfaces, which come in pairs!



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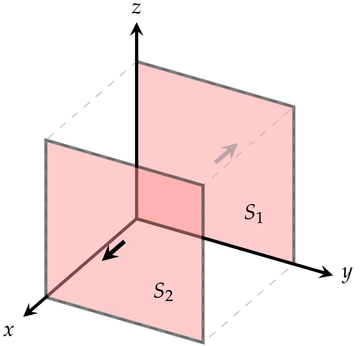
S_2 $x = a, 0 \leq y \leq a, 0 \leq z \leq a$



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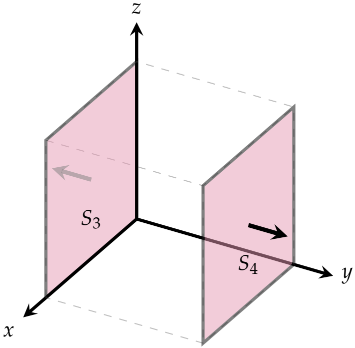
S_3 $y = 0, 0 \leq x \leq a, 0 \leq z \leq a$

S_4 $y = a, 0 \leq x \leq a, 0 \leq z \leq a$



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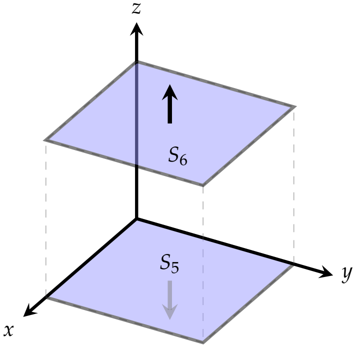
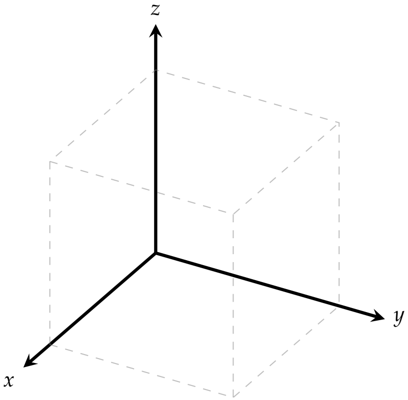
S_4 $y = a, 0 \le x \le a, 0 \le z \le a$

S_5 $z = 0, 0 \le x \le a, 0 \le y \le a$
S_6 $z = a, 0 \le x \le a, 0 \le y \le a$



The Flux of a Vector Field out of a Box

The flux of a vector field out of a box is a sum of six terms, one for each cube face.



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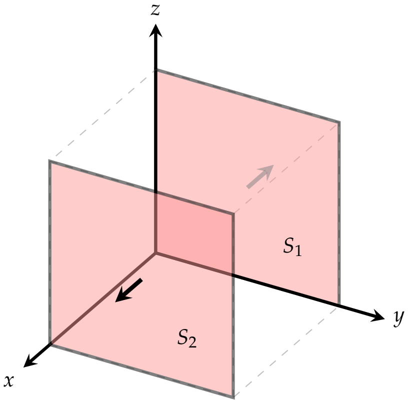
The flux of a vector field out of a box is a sum of six terms, one for each cube face.

$$\int_0^a \int_0^a P(a,y,z)\,dy\,dz -$$
$$\int_0^a \int_0^a P(0,y,z)\,dy\,dz +$$



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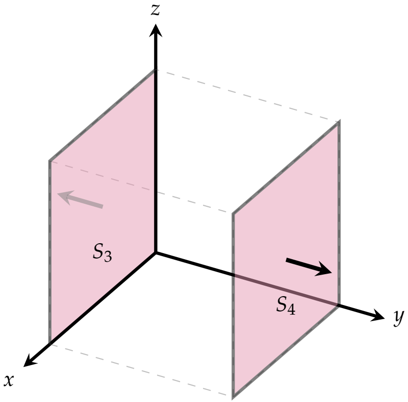
$$\int_0^a \int_0^a Q(x,0,z)\,dx\,dz$$



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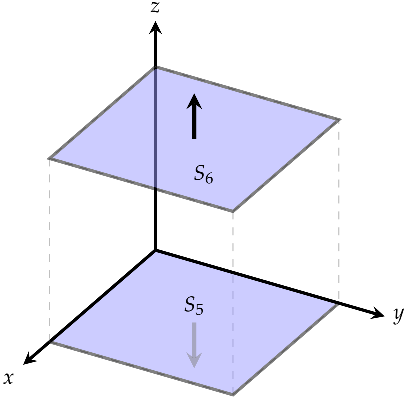
$$\int_0^a \int_0^a Q(x,a,z)\,dx\,dz -$$
$$\int_0^a \int_0^a Q(x,0,z)\,dx\,dz$$

$$\int_0^a \int_0^a R(x,y,a)\,dx\,dy -$$
$$\int_0^a \int_0^a R(x,y,0)\,dx\,dy +$$



The Gauss Divergence Theorem

Carl Friedrich Gauss, 1777-1855

Theorem Let E be a simple solid region and let S be a boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then Then

$$\iiint_E \nabla \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

where $\mathbf{n}$ is the outward unit normal to S.

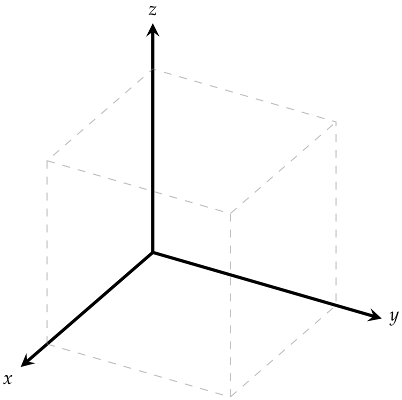
Important Note: the notations $\mathbf{F} \cdot \mathbf{n} \, dS$ (here) and $\mathbf{F} \cdot \, d\mathbf{S}$ (the book) mean the same thing.

The Divergence Theorem for a Cube

We can compute

$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV$$

on a cube of side a using the Fundamental Theorem of Calculus.



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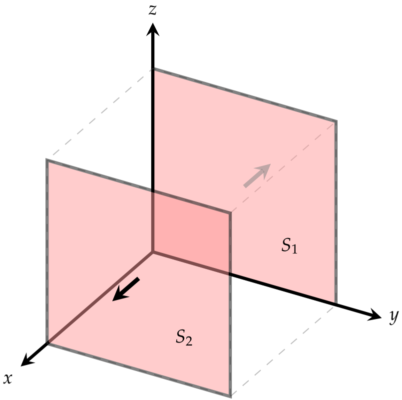
$$\int_0^a \int_0^a \int_0^a \frac{\partial P}{\partial x} \, dx \, dy \, dz$$
$$= \int_0^a \int_0^a \left(P(a,y,z) - P(0,y,z) \right) dy \, dz$$



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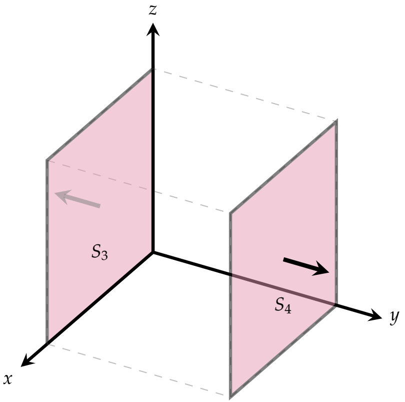
$$\int_0^a \int_0^a \int_0^a \frac{\partial P}{\partial x}\, dx\, dy\, dz$$
$$= \int_0^a \int_0^a (P(a,y,z) - P(0,y,z))\, dy\, dz$$

$$\int_0^a \int_0^a \int_0^a \frac{\partial Q}{\partial y}\, dy\, dx\, dz$$
$$= \int_0^a \int_0^a (Q(x,a,z) - Q(x,0,z))\, dx\, dz$$



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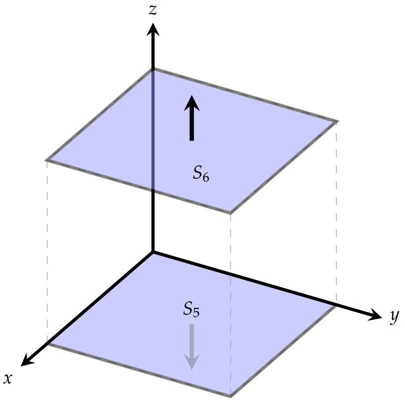
$$\int_0^a \int_0^a \int_0^a \frac{\partial Q}{\partial y}\, dy\, dx\, dz = \int_0^a \int_0^a (Q(x,a,z) - Q(x,0,z))\, dx\, dz$$

$$\int_0^a \int_0^a \int_0^a \frac{\partial R}{\partial z}\, dz\, dx\, dy = \int_0^a \int_0^a (R(x,y,a) - R(x,y,0))\, dx\, dy$$



Using the Divergence Theorem

Compute $\int_S \mathbf{F} \cdot d\mathbf{S}$ if

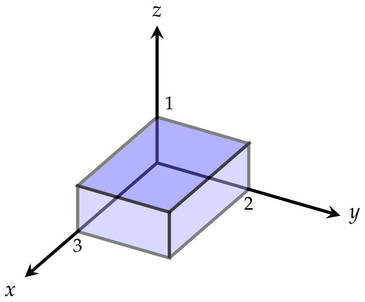
$$\mathbf{F}(x, y, z) = xye^z\mathbf{i} + xy^2z^3\mathbf{j} - ye^x\mathbf{k}$$

and S is the surface bounded by the coordinate planes and the planes $x = 3$, $y = 2$, and $z = 1$

Using the divergence theorem we can simply integrate div $\mathbf{F}$ over the region

$$\{0 \leq x \leq 3, 0 \leq y \leq 2, 0 \leq z \leq 1\}$$

Set up and compute this volume integral.



Using the Divergence Theorem

Divergence Theorem: If E is a simple closed surface and S is the oriented boundary of E, then

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ if

$$\mathbf{F} = (x^3 + y^3)\mathbf{i} + (y^3 + z^3)\mathbf{j} + (z^3 + x^3)\mathbf{k}$$

and S is the sphere of radius 2 with center at the origin.

1. Calculate div **F**
2. What's the easiest way to compute the volume integral of div **F** over the sphere of radius 2?

Vector Calculus Identities

Divergence Theorem: If E is a simple closed surface and S is the oriented boundary of E, then

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

Prove that if $\mathbf{a}$ is a constant, then $\iint_S \mathbf{a} \cdot d\mathbf{S} = 0$

Prove that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$ for a closed surface. (Hint: You can check that $\operatorname{div} \operatorname{curl} \mathbf{F} = 0$).

What We Learned About the Divergence

What does the divergence measure? From the divergence theorem we learn that div $\mathbf{F}$ measures *net outward flow per unit volume*, If E is a very small volume surrounded by a surface S, then

$$\iiint_E \operatorname{div} \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

$$\operatorname{div} \mathbf{F} \, \Delta V \simeq \iint_S \mathbf{F} \cdot d\mathbf{S}$$

So, for example if div $\mathbf{F} = 0$, this means that the net flux is zero, i.e., inflow $=$ outflow