



Math 213 - The Cross Product

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Reminders

- Access your WebWork account *only through Canvas!*
- Homework A2 on Sections 12.3-12.4 is due Friday
- Applications for an alternate Exam 1 are due **today!**

Review your schedule and apply for all alternate exams at once by using the [Google Form](#) linked from Canvas or the course home page.

Unit I: Geometry and Motion in Space

- 12.1 Lecture 1: Three-Dimensional Coordinate Systems
- 12.2 Lecture 2: Vectors in the Plane and in Space
- 12.3 Lecture 3: The Dot Product
- 12.4 **Lecture 4: The Cross Product**
- 12.5 Lecture 5: Equations of Lines and Planes, I
- 12.5 Lecture 6: Equations of Lines and Planes, II
- 12.6 Lecture 7: Surfaces in Space
- 13.1 Lecture 8: Vector Functions and Space Curves
- 13.2 Lecture 9 Derivatives and Integrals of Vector Functions
- Lecture 10: Exam I Review

Learning Goals

- Know how to compute determinants of orders 2 and 3
- Know how to compute the cross product $\mathbf{a} \times \mathbf{b}$ of two vectors using determinants and understand its geometric meaning
- Learn the properties of the cross product
- Understand the scalar triple product and its geometric meaning
- Understand how to use cross products to compute torque

Determinants

We'll need to know how to compute the *determinant* of a 2×2 or 3×3 matrix.

A **determinant of order 2** is defined by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Find the following determinants:

$$\begin{vmatrix} 2 & 1 \\ 4 & -6 \end{vmatrix}, \quad \begin{vmatrix} 4 & -6 \\ 2 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$$

Determinants, Continued

A **determinant of order 3** is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

For an illustration of this formula, see this [Khan Academy Video](#)

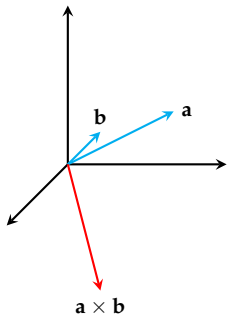
For a shortcut method that many students like, see this [Khan Academy Video](#)

Find, using your favorite method:

$$\begin{vmatrix} 1 & -4 & -2 \\ 2 & 0 & 4 \\ -1 & 2 & 3 \end{vmatrix}$$

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of \mathbf{a} and \mathbf{b} is the vector

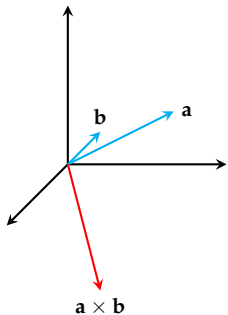
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



Find $\mathbf{a} \times \mathbf{b}$ if $\mathbf{a} = \langle 1, 0, 2 \rangle$ and $\mathbf{b} = \langle 1, 1, 1 \rangle$.

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of \mathbf{a} and \mathbf{b} is the vector

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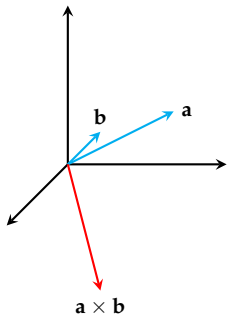


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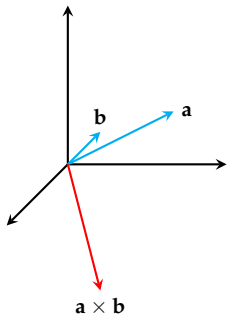


Find $\mathbf{a} \times \mathbf{b}$ if $\mathbf{a} = \langle 1, 0, 2 \rangle$ and $\mathbf{b} = \langle 1, 1, 1 \rangle$.

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \end{aligned}$$

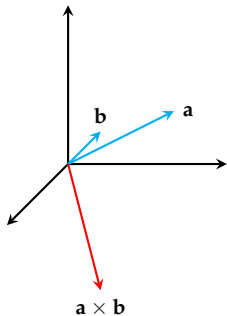
If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of \mathbf{a} and \mathbf{b} is the vector

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Find $\mathbf{a} \times \mathbf{b}$ if $\mathbf{a} = \langle 1, 0, 2 \rangle$ and $\mathbf{b} = \langle 1, 1, 1 \rangle$.

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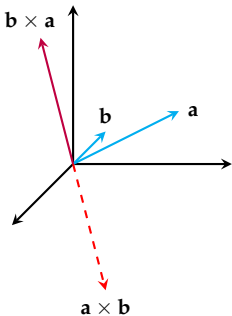
$$\mathbf{a} = \mathbf{i} + 0\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$$

What are $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ and $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b})$?

What is $\mathbf{b} \times \mathbf{a}$?



$$\mathbf{a} = \mathbf{i} + 0\mathbf{j} + 2\mathbf{k}$$

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What are $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ and $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b})$?

What is $\mathbf{b} \times \mathbf{a}$?

Cross Product Properties

- 1 The cross product $\mathbf{a} \times \mathbf{b}$ is perpendicular to \mathbf{a} and \mathbf{b} with direction given by the right-hand rule
- 2 $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram spanned by \mathbf{a} and \mathbf{b} , i.e.,

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$$

- 3 Two vectors \mathbf{a} and \mathbf{b} are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

- 1 Find $\mathbf{i} \times \mathbf{j}$, $\mathbf{i} \times \mathbf{k}$, and $\mathbf{j} \times \mathbf{k}$
- 2 Use this result to find $(2\mathbf{j} - 4\mathbf{k}) \times (-\mathbf{i} + 3\mathbf{j} + \mathbf{k})$
- 3 Which of the following expressions is meaningful?

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

$$\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$$

$$\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$$

$$(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$$

Cross Product Properties

Can you fill in the blanks?

If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors and if c is a scalar:

- ① $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- ② $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
- ③ $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- ④ $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
- ⑤ $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
- ⑥ $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

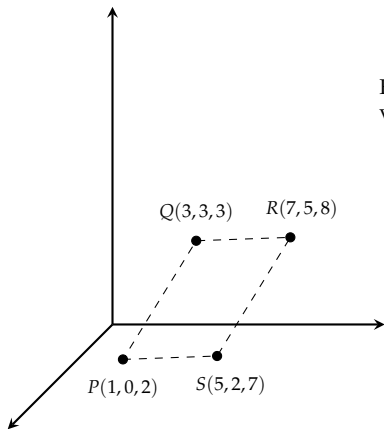
Cross Product Properties

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- ① $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
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- ③ $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \underline{\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}}$
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Cross Product Puzzlers

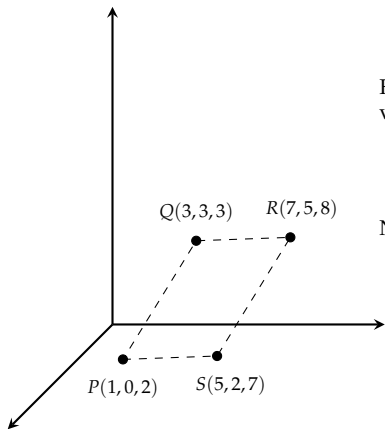


Find the area of the parallelogram with vertices

$$P(1, 0, 2), \quad Q(3, 3, 3),$$

$$R(7, 5, 8), \quad S(5, 2, 7).$$

Cross Product Puzzlers



Find the area of the parallelogram with vertices

$$P(1, 0, 2), \quad Q(3, 3, 3),$$

$$R(7, 5, 8), \quad S(5, 2, 7).$$

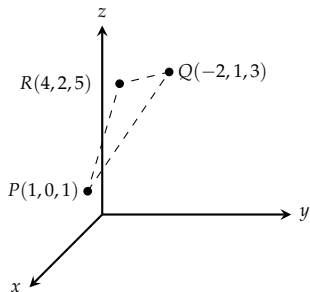
Note that:

$$\overrightarrow{PQ} = \langle 2, 3, 1 \rangle$$

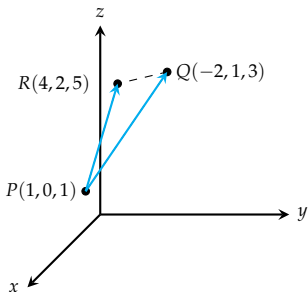
$$\overrightarrow{PS} = \langle 4, 2, 5 \rangle$$

Cross Product Puzzlers

Find a nonzero vector orthogonal to the plane through $P(1,0,1)$, $Q(-2,1,3)$, and $R(4,2,5)$ and find the area of triangle PQR



Cross Product Puzzlers



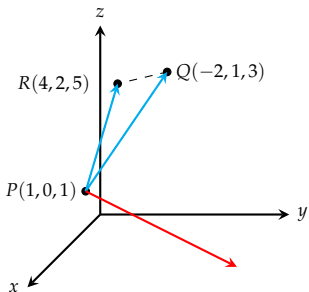
Find a nonzero vector orthogonal to the plane through $P(1,0,1)$, $Q(-2,1,3)$, and $R(4,2,5)$ and find the area of triangle PQR

Two sides of the triangle are spanned by:

$$\vec{PR} = \langle 3, 2, 4 \rangle$$

$$\vec{PQ} = \langle -3, 1, 2 \rangle$$

Cross Product Puzzlers



Find a nonzero vector orthogonal to the plane through $P(1,0,1)$, $Q(-2,1,3)$, and $R(4,2,5)$ and find the area of triangle PQR

Two sides of the triangle are spanned by:

$$\vec{PR} = \langle 3, 2, 4 \rangle$$

$$\vec{PQ} = \langle -3, 1, 2 \rangle$$

The cross product is:

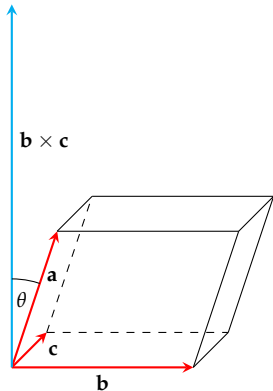
$$\vec{PQ} \times \vec{PR} = \langle 0, 18, -9 \rangle$$

(not drawn to scale!)

Scalar Triple Product

The *scalar triple product* of three vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} is the determinant

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

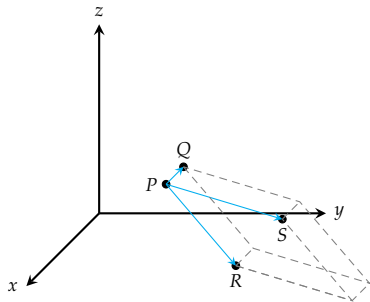


The volume of the parallelepiped formed by the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} is given by

$$|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$

What happens if the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} are coplanar?

Triple Product Puzzler

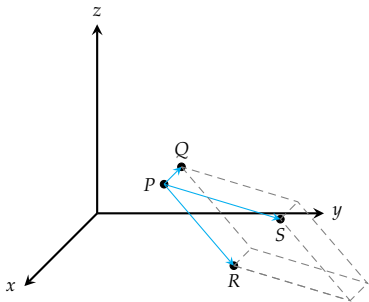


Find the volume of the parallelepiped with adjacent edges PQ , PR , and PS if

$$P = P(-2, 1, 0), \quad Q = Q(2, 3, 2),$$

$$R = R(1, 4, -1) \quad S = S(3, 6, 1)$$

Triple Product Puzzler



Find the volume of the parallelepiped with adjacent edges PQ , PR , and PS if

$$P = P(-2, 1, 0), \quad Q = Q(2, 3, 2),$$

$$R = R(1, 4, -1) \quad S = S(3, 6, 1)$$

$$\overrightarrow{PQ} = \langle 4, 2, 2 \rangle$$

$$\overrightarrow{PR} = \langle 3, 3, -1 \rangle$$

$$\overrightarrow{PS} = \langle 5, 5, 1 \rangle$$

Torque

If a force \mathbf{F} acts on a rigid body at a position \mathbf{r} , the **torque** $\boldsymbol{\tau}$ is given by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

The torque vector points along the axis of rotation

Two friends sit on a see-saw. Friend 1 weighs 100 N and sits 2 m to the left of the fulcrum, while friend 2 was 50N and sits 5 m to the right of the fulcrum. Find the torques and predict which way the see-saw will go.

Dot Product, Cross Product, Triple Product

	Dot Product	Cross Product	Scalar Triple Product
Type	Scalar $\mathbf{a} \cdot \mathbf{b}$	Vector $\mathbf{a} \times \mathbf{b}$	Scalar $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
Magnitude	$ \mathbf{a} \mathbf{b} \cos \theta$	$ \mathbf{a} \mathbf{b} \sin \theta$	
Symmetry	$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$	$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$	Antisymmetric
Direction	None!	Right-hand rule	None!
In Physics	Work	Torque	
In Geometry	Projection	Parallelogram Area	Parallelepiped Area

Lecture Review

- We reviewed rules for computing 2×2 and 3×3 determinants
- We defined the *cross product* of two vectors, found its properties, and found its geometric meaning (right hand rule, area of parallelogram)
- We defined the *scalar triple product* of three vectors and found its geometric meaning (signed volume of a parallelepiped)
- We reviewed basics of the dot product (scalar) and the cross product (vector)

Homework

- Webwork A2 is due Friday
- Re-read section 12.4, pp. 814–821
- Prepare for your recitation and study for Quiz #1 on sections 12.1–12.2
- Remember to access WebWork *only through Canvas!*

Also, read section 12.5, pp. 823–830 for Friday. We'll spend two lectures on this material