Math 213 - Quadric Surfaces

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University of Kentucky

September 11, 2019

Reminders

- Access your WebWork account *only through Canvas*!
- Homework A3 on section 12.5 is due tonight
- Homework A4 on section 12.6 is due Friday
- Quiz # 2 on sections 12.3-12.4 takes place in tomorrow's recitation

Unit I: Geometry and Motion in Space

12.1 Lecture 1: Three-Dimensional Coordinate Systems

12.2 Lecture 2: Vectors in the Plane and in Space

12.3 Lecture 3:The Dot Product

12.4 Lecture 4:The Cross Product

12.5 Lecture 5: Equations of Lines and Planes, I

12.5 Lecture 6: Equations of Lines and Planes, II

12.6 Lecture 7: Surfaces in Space

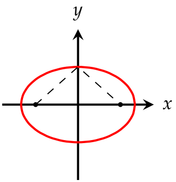
13.1 Lecture 8: Vector Functions and Space Curves

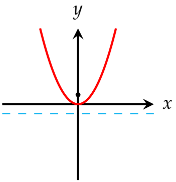
13.2 Lecture 9 Derivatives and Integrals of Vector Functions

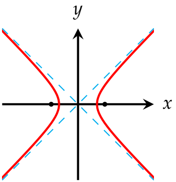
Lecture 10: Exam I Review

Learning Goals

- Recall quadratic curves from MA 114
- Identify and graph cylinders
- Find traces of quadric surfaces
- Identify and graph quadric surfaces





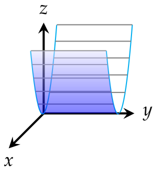


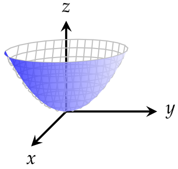
A *quadratic curve* is the graph of a second-degree equation in two variables taking one of the forms

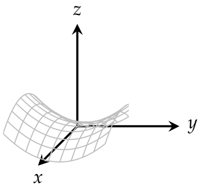
$$Ax^2 + By^2 + J = 0, \quad Ax^2 + By^2 + Jy = 0$$

In MA 114 you learned about the following *quadratic curves*:

- The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with foci $(\pm c, 0)$, where $a^2 = b^2 + c^2$
- The parabola $x^2 = 4py$ with focus at $(0, p)$ and directrix at $y = -p$
- The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with foci at $(\pm c, 0)$ where $c^2 = a^2 + b^2$





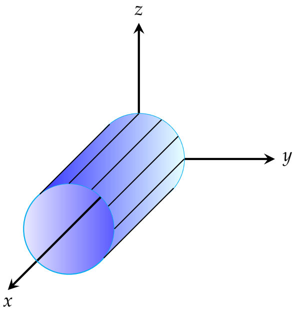


In MA 213 we'll study *quadric surfaces* in three dimensional space, such as:

- *Cylinders* which consist of all lines that are parallel to a given line and pass through a given plane curve
- *Elliptic Paraboloids* which will model functions with local maxima or minima
- *Hyperbolic Paraboloids* ("saddles") which model a new kind of critical point, called a *saddle point*, for functions of two variables

Cylinders

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

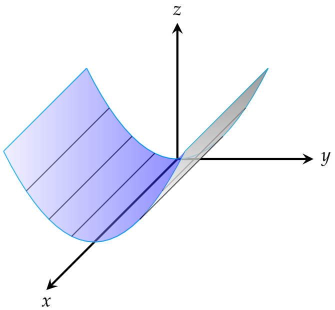


Example 1: $y^2 + z^2 = 1$

- What is the given curve?
- What is the given line?

Cylinders

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.



Example 2: $z = y^2$

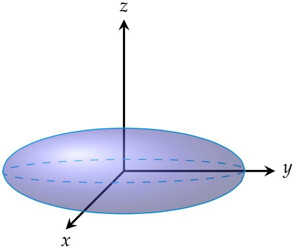
- What is the given curve?
- What is the given line?

Quadric Surfaces

A **quadric surface** is the graph of a second-degree equation in x, y, and z taking one of the standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0, \quad Ax^2 + By^2 + Iz = 0$$

We can graph a quadric surface by studying its *traces* in planes parallel to the x, y, and z axes. The traces are always quadratic curves!



In what follows we'll use the method of traces to graph the *ellipsoid*

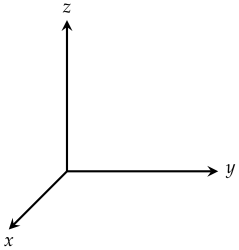
$$x^2 + \frac{y^2}{4} + \frac{z^2}{2} = 1$$

Interlude: A Buddhist Parable

A group of blind men heard that a strange animal, called an elephant, had been brought to the town, but none of them were aware of its shape and form. Out of curiosity, they said: "We must inspect and know it by touch, of which we are capable". So, they sought it out, and when they found it they groped about it. In the case of the first person, whose hand landed on the trunk, said "This being is like a thick snake". For another one whose hand reached its ear, it seemed like a kind of fan. As for another person, whose hand was upon its leg, said, the elephant is a pillar like a tree-trunk. The blind man who placed his hand upon its side said, "elephant is a wall". Another who felt its tail, described it as a rope. The last felt its tusk, stating the elephant is that which is hard, smooth and like a spear.

Graphing the Ellipsoid by Traces

$$x^2 + \frac{y^2}{4} + \frac{z^2}{2} = 1$$

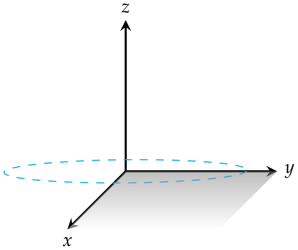


Graphing the Ellipsoid by Traces

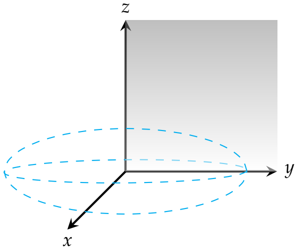
$$x^2 + \frac{y^2}{4} + \frac{z^2}{2} = 1$$

Trace in the xy-plane ($z = 0$):

$$x^2 + \frac{y^2}{4} = 1$$



Graphing the Ellipsoid by Traces



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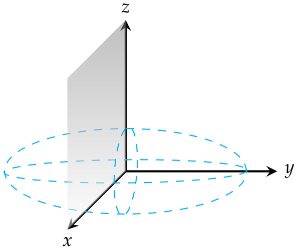
Trace in the xy-plane ($z = 0$):

$$x^2 + \frac{y^2}{4} = 1$$

Trace in the yz-plane ($x = 0$):

$$\frac{y^2}{4} + \frac{z^2}{2} = 1$$

Graphing the Ellipsoid by Traces



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Trace in the xz-plane ($y = 0$):

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Graphing the Ellipsoid by Traces

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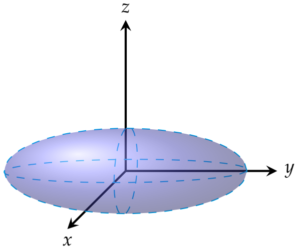

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Trace in the yz-plane ($x = 0$):

$$\frac{y^2}{4} + \frac{z^2}{2} = 1$$

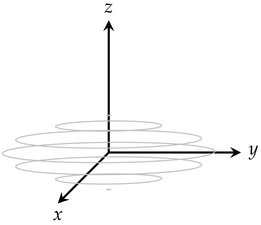
Trace in the xz-plane ($y = 0$):

$$x^2 + \frac{z^2}{2} = 0$$

More on Traces

It often helps to find the traces in planes *parallel* to the xy, xz, or yz planes.

$$x^2 + \frac{y^2}{4} + \frac{z^2}{2} = 1$$



Trace in planes $z = k$:

$$x^2 + \frac{y^2}{4} + \frac{k^2}{2} = 1$$

$$x^2 + \frac{y^2}{4} = 1 - \frac{k^2}{2}$$

$$\frac{x^2}{1 - k^2/2} + \frac{y^2}{4 - 2k^2} = 1$$

- Traces are ellipses
- No trace if $|k| > \sqrt{2}$

More on Traces

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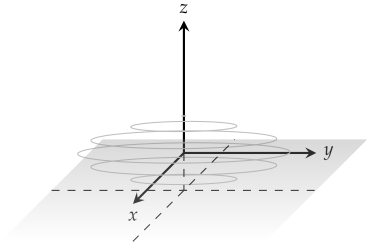

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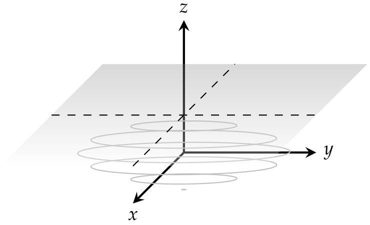

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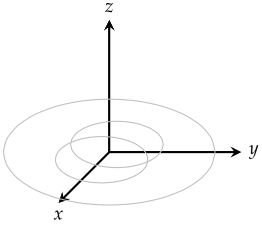

Trace in planes $x = k$:

$$\begin{aligned} k^2 + \frac{y^2}{4} + \frac{z^2}{2} &= 1 \\ \frac{y^2}{4} + \frac{z^2}{2} &= 1 - k^2 \\ \frac{y^2}{4 - 4k^2} + \frac{z^2}{2 - 2k^2} &= 1 \end{aligned}$$

- Traces are ellipses
- No trace if $|k| > 1$

More on Traces

It often helps to find the traces in planes *parallel* to the xy, xz, or yz planes.

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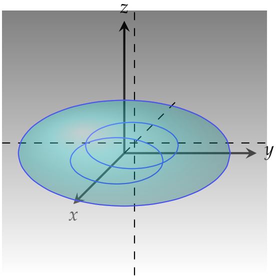

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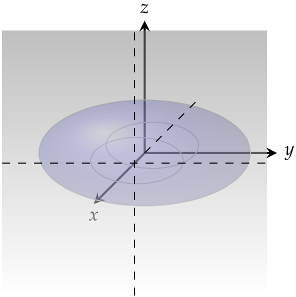

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- Traces are ellipses
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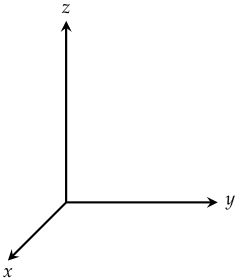
Elliptic Paraboloid

We'll use the method of traces to graph the surface

$$z = x^2 + y^2$$



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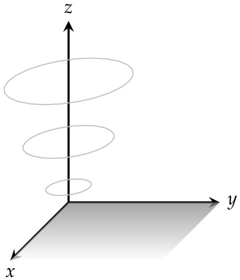


Traces in planes $z = k$:

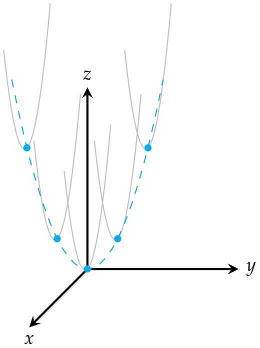
$$x^2 + y^2 = k$$

- No traces below $z = 0$
- Traces are circles of radius $\sqrt{k}$ and center $(0, 0, k)$

Elliptic Paraboloid

We'll use the method of traces to graph the surface

$$z = x^2 + y^2$$



Traces in planes $y = k$:

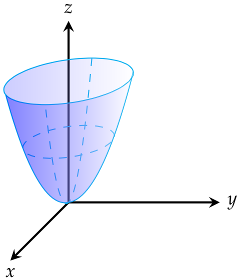
$$x^2 + k^2 = z$$

- No restriction on traces
- Traces are upward parabolas with vertex $(0, k, k^2)$

Elliptic Paraboloid

We'll use the method of traces to graph the surface

$$z = x^2 + y^2$$



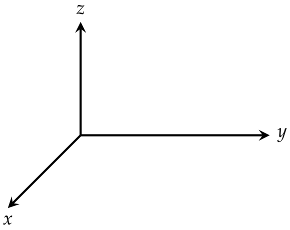
Putting it all together. . .

- Traces parallel to xy plane are circles
- Traces parallel yz or xz plane are parabolas

Hyperbolic Paraboloid (Saddle)

We'll use the method of traces to graph the surface

$$z = y^2 - x^2$$



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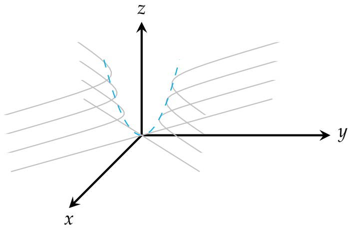


Traces in plane $z = k$:

$$y^2 - x^2 = k$$

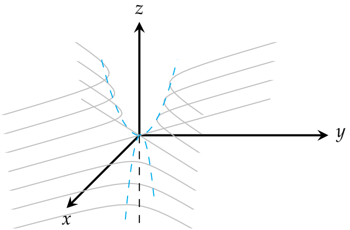
Hyperbolas:

- Opening in the y direction, $k > 0$

Hyperbolic Paraboloid (Saddle)

We'll use the method of traces to graph the surface

$$z = y^2 - x^2$$



Traces in plane $z = k$:

$$y^2 - x^2 = k$$

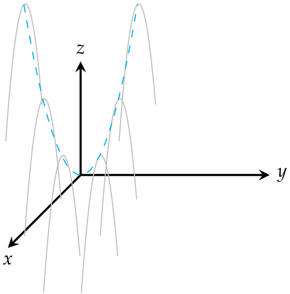
Hyperbolas:

- Opening in the y direction, $k > 0$
- Opening in the x direction, $k < 0$

Hyperbolic Paraboloid (Saddle)

We'll use the method of traces to graph the surface

$$z = y^2 - x^2$$



Traces in plane $y = k$:

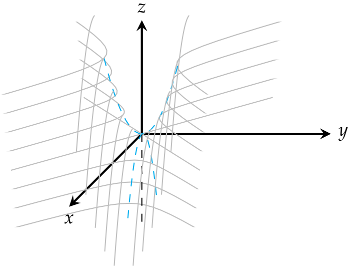
$$z = k^2 - x^2$$

Downward parabolas, vertex at $(k^2, 0)$

Hyperbolic Paraboloid (Saddle)

We'll use the method of traces to graph the surface

$$z = y^2 - x^2$$

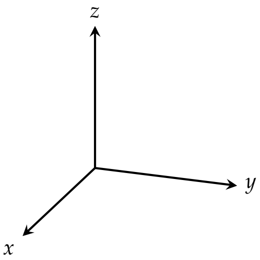


Putting it all together...

- Traces parallel to xy plane are hyperbolas
- Traces parallel to xz plane are downward parabolas

Hyperboloid of One Sheet

We'll use the method of traces to graph the surface $x^2 + y^2 - z^2 = 1$.



Rewrite the equation as:

$$x^2 + y^2 = 1 + z^2$$

What are the traces in the plane $z = k$?

Hyperboloid of One Sheet

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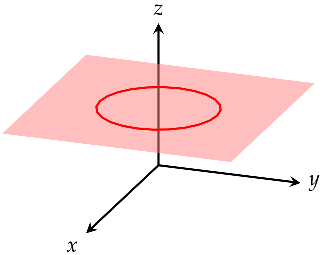


Rewrite the equation as:

$$x^2 + y^2 = 1 + z^2$$

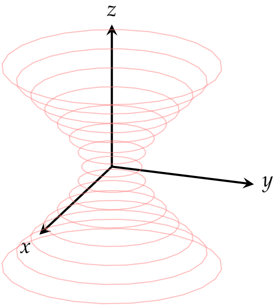
What are the traces in the plane $z = k$?

Equation: $x^2 + y^2 = 1 + k^2$

Curve: Circles with center at $(0, 0, k)$ of radius $\sqrt{1 + k^2}$

Hyperboloid of One Sheet

We'll use the method of traces to graph the surface $x^2 + y^2 - z^2 = 1$.



Rewrite the equation as:

$$x^2 + y^2 = 1 + z^2$$

What are the traces in the plane $z = k$?

Equation: $x^2 + y^2 = 1 + k^2$

Curve: Circles with center at $(0, 0, k)$ of radius $\sqrt{1 + k^2}$

Surface: *hyperboloid of one sheet* whose axis of symmetry is the z axis

Summary

- We recalled the equations and graphs of ellipses, parabolas, and hyperbolas from MA 114
- We learned how to graph equations of cylinders like $z = y^2$ or $x^2 + z^2 = 1$
- We learned how to find *traces* of quadric surfaces by restricting to planes $x = h$, $y = k$, or $z = \ell$
- We learned how to visualize a quadric surface given its traces
- We discussed four important quadric surfaces:
  - The ellipsoid (example: $x^2/9 + y^2/4 + z^2 = 1$)
  - The elliptic paraboloid (example: $z = x^2 + y^2$)
  - The "saddle" (example: $z = x^2 - y^2$)
  - The hyperboloid of one sheet (example: $x^2 + y^2 - z^2 = 1$)

Homework

- Re-read section 12.6 and continue work on webwork A4 (due Friday)
- Read and study section 13.1 on vector functions and space curves
- Prepare for tomorrow's recitation on section 12.6
- Prepare for tomorrow's quiz on sections 12.3-12.4