Math 213 - Motion in Space

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Homework

- Your first exam is Wednesday, February 6 at 5:00 PM in CB 106
- There will be an Exam I Review Session tonight, 6:00-8:00 PM in room CP 139
- Re-read section 13.3-4 (in section 13.3, omit curvature, normal, binormal vectors)
- Work on Stewart problems:
  13.3: 1, 3, 5, 11, 13, 17, 19 (odd)
  13.4: 3, 7, 9, 11, 23, 25

Unit I: Geometry and Motion in Space (Revised)

Goals of the Day

- Know how to compute velocity and acceleration
- Know how to solve projectile problems
- Understand how to compute arc length

Velocity and Acceleration

If $\mathbf{r}(t)$ is the space curve of a moving body and if t is time:

- $\mathbf{r}'(t)$ is $\mathbf{v}(t)$, the *velocity* of the moving body
- $|\mathbf{r}'(t)|$ is the *speed* of the moving body
- $\mathbf{r}''(t)$ is $\mathbf{a}(t)$, the *acceleration* of the moving body

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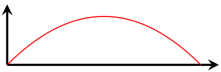
1. (Projectile motion) Suppose that $\mathbf{r}(t) = \langle 32t, 32t - 16t^2 \rangle$. Find the velocity and acceleration
2. (Circular motion) Suppose that $\mathbf{r}(t) = \langle R\cos(2\pi t/T), R\sin(2\pi t/T) \rangle$. Find the velocity and acceleration.

Velocity and Acceleration



$\mathbf{r}(t) = \langle 32t, 32t - 16t^2 \rangle.$

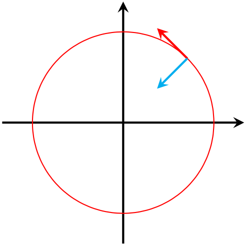
What's the projectile's acceleration?

When does the projectile hit the ground?

What is its speed when it hits?

How far does it go?

What is its maximum height?



$\mathbf{r}(t) = \langle R\cos(2\pi t/T), R\sin(2\pi t/T) \rangle$

How long does one orbit take?

Which way does the velocity vector point?

Which way does the acceleration vector point?

Math 114 Reminder

In Math 114, we defined the arc length of a parameterized curve

$$x = f(t), \quad y = g(t), \quad a \leq t \leq b$$

as

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

We can now recognize arc length as *the integral of speed*: if

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$$

then the velocity along the curve is

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$$

and the speed is

$$|\mathbf{r}'(t)| = \sqrt{f'(t)^2 + g'(t)^2}$$

Arc Length

For a space curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, the arc length of the curve between $t = a$ and $t = b$ is:

$$
\begin{aligned}
L &= \int_a^b \left|\mathbf{r}'(t)\right| \, dt \\
&= \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt
\end{aligned}
$$

Find the arc length of the curve

$$\mathbf{r}(t) = \langle cos(t)\mathbf{i} + \sin(t)\mathbf{j} + ln(\cos t)\mathbf{k}$$

between $t = 0$ and $t = \pi/4$.

Interlude - Newton's Laws of Motion

1. A body will remain at rest or in motion in a straight line unless acted on by an external force.
2. The applied force $\mathbf{F}$ is equal to the change of momentum $m\mathbf{v}$ per unit time
3. For every action there is an equal and opposite reaction

Projectile Motion

For constant mass, Newton's second law implies

$$\mathbf{F} = m\mathbf{a}$$

(Warning: Do not use for rockets!)

At the surface of the earth, a mass m is subject to a gravitational force $-mg\mathbf{k}$

From Newton's second law we then get $m\mathbf{a} = -mg\mathbf{j}$ or

$$\mathbf{r}''(t) = \mathbf{a} = -g\mathbf{k}$$

where $g = 32$ ft/sec$^2 = 9.8$ m/sec^2.

If we know the *initial conditions* for a projectile (its position and velocity at time zero), we can integrate this equation to find the motion of the projectile.

Projectile Motion - Metric Units

A ball is thrown at an angle of 45° to the ground. If the ball lands 90 m away, what was the initial speed of the ball?

$$\begin{aligned} \mathbf{r}''(t) &= -9.8\mathbf{k} \\ \mathbf{r}'(0) &= \mathbf{v}(0) = v_0 \cos(45^\circ)\mathbf{i} + v_0 \sin(45^\circ)\mathbf{k} \\ \mathbf{r}(0) &= 0\mathbf{i} + 0\mathbf{k} \end{aligned}$$

Now integrate:

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{v}(0) + \int_0^t \mathbf{a}(s)\, ds \\ &= v_0(\sqrt{2}/2)\mathbf{i} + \left(v_0(\sqrt{2}/2) - 9.8t\right)\mathbf{k} \\ \mathbf{r}(t) &= \mathbf{r}(0) + (v_0(\sqrt{2}/2)t)\mathbf{i} + \left(v_0(\sqrt{2}/2)t - (9.8/2)t^2\right)\mathbf{k} \end{aligned}$$

Now what?

More Fun with Projectile Motion - English Units

A rifle is fired with angle of elevation 36°. What is the muzzle speed if the maximum height of the bullet is 1600 ft?

Yet More Fun with Projectile Motion

A batter hits a baseball 3ft above the ground toward the center field fence, which is 10 ft high and 400 ft from home plate. The ball leaves the bat with speed 115 ft/sec at an angle of 50° above the horizontal. It is a home run? (that is, does the ball clear the fence?)

Projectile Motion - Some Takeways

Given the position function

$$\mathbf{r}(t) = x(t)\mathbf{i} + z(t)\mathbf{k}$$

for a projectile, how do you determine...

- The maximum height of the projectile?
  (At what time t does this occur?)
- The range of the projectile?
  (At what time t does the projectile hit the ground?)
- The speed of the projectile at impact?

Summary

We discussed:

- How to find the velocity, speed, and acceleration from the vector function $\mathbf{r}(t)$ that describes the motion of a particle in space
- How to compute arc length by integrating the speed
- How to solve projectile problems