Math 213 - Directional Derivatives and the Gradient Vector

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Homework

- Remember that WebWork B4 is due Wednesday
- Start working on practice problems in section 14.6, 11-23 (odd), 29, 31, 33, 42, 43, 54, 55, 57
- Re-read section 14.6
- Read ahead for Wednesday - section 14.7

Unit II: Differential Calculus of Several Variables

Goals of the Day

- Understand what a *directional derivative* of a function of two and three variables is, and how to compute it
- Understand what the *gradient vector* ∇f is, and how it's related to directional derivatives
- Understand that the gradient vector:
  - Points in the direction of maximum change of f
  - Has magnitude equal to that maximal rate of change
  - Is perpendicular to level curves (two variables) or level surfaces (three variables)

Review

The Chain Rule, 2 Variables (Case 1) If

$$z = f(x, y),\ x = g(t),\ y = h(t),$$

then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Suppose that

$$f_x(1,3) = -2, \quad f_y(1,3) = 4,$$
$$x(t) = t, \quad y(t) = 3t.$$

What is the derivative of $f(x(t), y(t))$ at $t = 1$?

Review

The Chain Rule, 2 Variables (Case 1) If

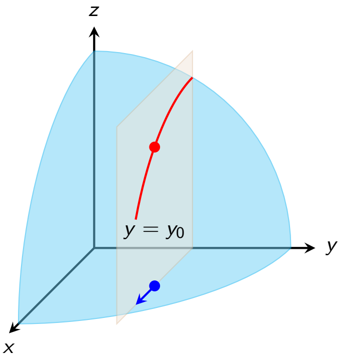
$$z = f(x, y),\ x = g(t),\ y = h(t),$$

then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

If $x(t) = x_0 + at$, $y(t) = y_0 + bt$, what is $(d/dt)f(x(t), y(t))$ at $t = 0$?

Review

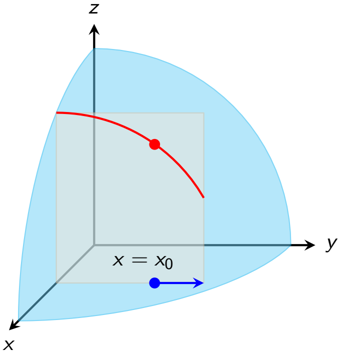


Suppose $f(x,y)$ is a function of two variables, and (x_0,y_0) is a point in its domain. The partial derivatives of f with respect to x and y are given by:

$$f_x(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

(change in direction of $\langle 1,0\rangle$)

Review



Suppose $f(x, y)$ is a function of two variables, and (x_0, y_0) is a point in its domain. The partial derivatives of f with respect to x and y are given by:

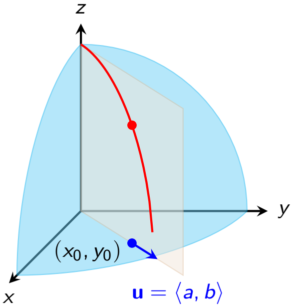
$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

(change in direction of $\langle 1, 0 \rangle$)

$$f_y(x_0, , y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

(change in direction of $\langle 0, 1 \rangle$)

The Directional Derivative



Suppose $f(x, y)$ is a function of two variables, (x_0, y_0) is a point in its domain, and $\mathbf{u} = \langle a, b \rangle$ is a unit vector.

The directional derivative of f in the direction $u = \langle a, b \rangle$ at (x_0, y_0) is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

Computing the Directional Derivative

Remember that

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

If $\mathbf{u} = \langle a, b \rangle$, this is the same as the derivative of $f(x_0 + at, y_0 + bt)$ at $t = 0$.

We can compute this by the chain rule and get

$$D_{\mathbf{u}}f(x_0, y_0) = af_x(x_0, y_0) + bf_y(x_0, y_0)$$

1. Find the directional derivative of $f(x, y) = xy^3 - x^2$ at $(1, 2)$ in the direction $\mathbf{u} = \langle 1/2, \sqrt{3}2$
2. Find the directional derivative of $f(x, y) = x^2 \ln y$ at $(3, 1)$ in the direction of $\mathbf{u} = (-5/13)\mathbf{i} + (12/13)\mathbf{j}$

The Gradient Vector

We can look at the formula

$$D_{\mathbf{u}}f(x_0, y_0) = af_x(x_0, y_0) + bf_y(x_0, y_0)$$

in a new way by introducing the *gradient of f at* (x_0, y_0): if

$$(\nabla f)(x_0, y_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$$

then

$$D_{\mathbf{u}}f(x_0, y_0) = \nabla f \cdot \mathbf{u}$$

1. Find the gradient vector for $f(x, y) = x/y$ at $(2, 1)$
2. Find the directional derivative of f at $(2, 1)$ in the direction $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

Maximum Rate of Change

Remember that

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos(\theta)$$

where θ is the angle between **a** and **b**.

The dot product has its maximum value when the vector **a** points in the same direction as **b**.

So, the directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ has its maximum when **u** points in the same direction as $\nabla f(x_0, y_0)$. In this direction, $D_{\mathbf{u}}f(x_0, y_0) = |\nabla f(x_0, y_0)|$

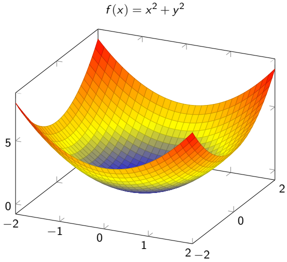
1. Find the maximum rate of change of $f(x, y) = xe^{xy}$ at $(0, 2)$ and find the direction where it occurs.

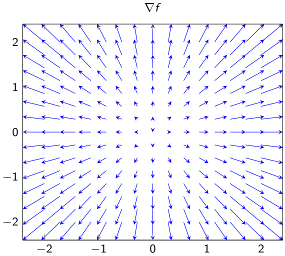
Sneak Preview: The Gradient Vector Field

If $f(x, y)$ is a function two variables, the *gradient vector field*

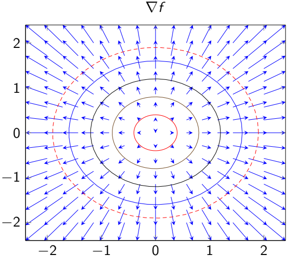
$$\nabla f(x, y) = \frac{\partial f}{\partial x}(x, y)\mathbf{i} + \frac{\partial f}{\partial y(x, y)}\mathbf{j}$$

moves in the direction of greatest change of f





The Gradient and Level Curves



The dot product of two vectors **a** and **b** is zero when **a** and **b** are perpendicular.

The directional derivative $D_{\mathbf{u}}f$ must be zero when **u** points along a level curve.

So, the gradient of a function f must be perpendicular to the level curves of f

Summary

- The *gradient* of a function $f(x, y)$ at $(x, y) = (x_0, y_0)$ is the vector

$$\nabla f(x_0, y_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$$

- If $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is a unit vector, then the directional derivative of f at (x_0, y_0) in the direction $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}$$

- The gradient $\nabla f(x_0, y_0)$ points in the direction of greatest change of f at (x_0, y_0). The magnitude of the gradient is equal to the greatest change.
- The gradient $\nabla f(x_0, y_0)$ is perpendicular to the level curve of f passing through (x_0, y_0)

Directional Derivatives of $f(x, y, z)$

The directional derivative of $f(x, y, z)$ at (x_0, y_0, z_0) in the direction $\mathbf{u} = \langle a, b, c \rangle$ is given by

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh, z_0 + ch) - f(x_0, y_0, z_0)}{h}$$

To compute it, we introduce the gradient vector

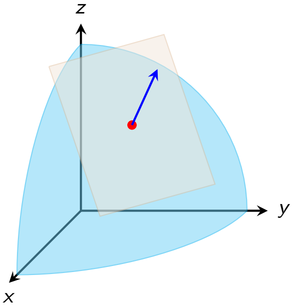
$$\nabla f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)\mathbf{i} + f_y(x_0, y_0, z_0)\mathbf{j} + f_z(x_0, y_0, z_0)\mathbf{k}$$

Then, if $\mathbf{u} = \langle a, b, c \rangle$ is a unit vector,

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \mathbf{u}.$$

1. Using the gradient vector, find the directional derivative of $f(x, y, z) = y^2 e^{xyz}$ at $(0, 1, -1)$ in the direction $\mathbf{u} = \frac{3}{13}\mathbf{i} + \frac{4}{13}\mathbf{j} + \frac{12}{13}\mathbf{k}$.
2. Find the maximum rate of change of $f(xy, z) = x \ln(yz)$ at $(1, 2, 1/2)$ and find the direction in which it occurs.

Tangent Planes to Level Surfaces



The gradient of a function of two variables is perpendicular to level curves of that function.

The gradient of a function of three variables is perpendicular to level surfaces of that function.

This means that $\nabla f(x_0, y_0, z_0)$ is normal to the tangent plane to f at (x_0, y_0, z_0)

1. Find the equations of the tangent plane and the normal line to the surface $x = y^2 + z^2 + 1$ at $(3, 1, -1)$.
2. Are there any points on the hyperboloid $x^2 - y^2 = -z^2 = 1$ where the tangent plane is parallel to the plane $z = x + y$?