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# Math 213 - Double Integrals in Polar Coordinates

### Peter A. Perry

University of Kentucky

March 4, 2019

### Homework

- Exam II Review Session is tonight in CP 139, 6-8 PM
- Exam II takes place this Wednesday in CB 106, 5-7 PM and will cover 14.1, 14.3-14.8, 15.1-15.2
- Webwork B8 on 15.1-15.2 is due Friday March 8
- Practice problems for 15.3 are 1-4, 5-31 (odd), 35, 37
- Webwork C1 on 15.3 will be due Friday March 8

Volumes

# Unit II: Differential Calculus of Several Variables

- Lecture 12 Functions of Several Variables
- Lecture 13 Partial Derivatives
- Lecture 14 Tangent Planes and Linear Approximation
- Lecture 15 The Chain Rule
- Lecture 16 Directional Derivatives and the Gradient
- Lecture 17 Maximum and Minimum Values, I
- Lecture 18 Maximum and Minimum Values, II
- Lecture 19 Lagrange Multipliers
- Lecture 20 Double Integrals
- Lecture 21 Double Integrals over General Regions
- Lecture 22 Double Integrals in Polar Coordinates

Lecture 23 Exam II Review

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### Goals of the Day

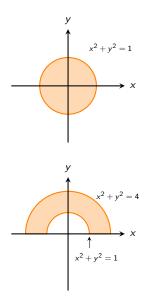
- Review Polar Coordinates, introduce Polar Rectangles
- Learn how to compute double integrals over polar rectangles
- Learn how to compute double integrals over polar regions
- Learn to compute volumes using polar integrals

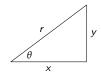
### **Reality Check**

	Calculus I	Calculus III
Riemann sum	$\sum_{i=1}^{n} f(x_i^*) \Delta x$	$\sum_{i,j=1}^{n} f(x_i^*, y_j^*) \Delta A$
Riemann Integral	$\int_{a}^{b} f(x)  dx$	$\iint_D f(x,y)  dA$
Way of computing	F(b) - F(a)	Iterated Integral
Interpretation	Area under a curve	Volume under a surface

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### Review of Polar Coordinates





Recall that

$$r^2 = x^2 + y^2$$
,  $\tan \theta = \frac{y}{x}$ 

and

$$x = r \cos \theta$$
,  $y = r \sin \theta$ .

How would you describe the regions at left in polar coordinates?

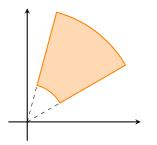
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### **Polar Rectangles**

A polar rectangle is a region

$$R = \{(r, \theta) : a \le r \le b, \quad \alpha \le \theta \le \beta\}.$$



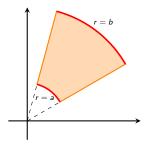
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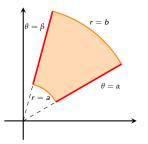


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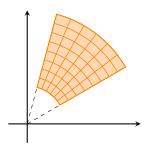
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### Polar Rectangles

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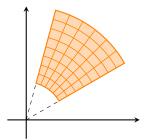
A small polar rectangle has area

 $\Delta A \simeq r \, \Delta r \, \Delta \theta$ 



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### Integrals Over Polar Rectangles

The double integral  $\iint_R f(x, y) dA$  is a limit of Riemann sums:

$$\sum_{i,j=1}^{n} f(r_i^* \cos \theta_j^*, r_i^* \sin \theta_j^*) r_j \, \Delta r \, \Delta \theta$$

Rectangle  $R_{ij}$  is given by

$$R_{ij} = \{(r, \theta) : r_{i-1} \le r \le r_i, \theta_{j-1} \le \theta \le \theta_j\}$$

$$r_i = \mathbf{a} + i\Delta r$$
 ,  $heta_j = lpha + j\Delta heta$ 

where

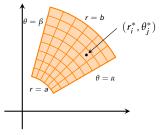
$$\Delta r = \frac{b-a}{n}, \quad \Delta \theta = \frac{\beta-\alpha}{n}$$

In the limit this leads to an iterated integral

$$\int_{\alpha}^{\beta} \int_{a}^{b} f(r\cos\theta, r\sin\theta) \, r \, dr \, d\theta$$

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## Integrals Over Polar Rectangles

**Double Integral In Polar Coordinates** The integral of a continuous function f(x, y) over a polar rectangle R given by  $a \le r \le b$ ,  $\alpha \le r \le \beta$ , is  $\iint_{R} f(x, y) \, dA = \int_{a}^{\beta} \int_{a}^{b} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta$ 

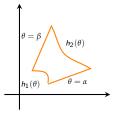
1. Find  $\iint_R (2x - y) dA$  if R is the region in the first quadrant bounded by the circle  $x^2 + y^2 = 4$  and the lines x = 0 and y = x.

2. Find 
$$\iint_R e^{-x^2-y^2} dA$$
 if D is the region bounded by the semicircle  $x = \sqrt{4-y^2}$  and the y-axis.

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## Integrals over Polar Regions



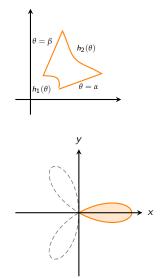
If f is continuous over a polar region of the form

$$D = \{(r, \theta) : \alpha \le \theta \le \beta, \ h_1(\theta) \le r \le h_2(\theta)\}$$

then

$$\iint_{D} f(x, y) dA =$$
$$\int_{\alpha}^{\beta} \int_{h_{1}(\theta)}^{h_{2}(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$

### Integrals over Polar Regions



If f is continuous over a polar region of the form

$$D = \{(r, \theta) : \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

then

$$\iint_{D} f(x, y) \, dA =$$
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Find the area of one loop of the rose

$$r = \cos 3\theta$$

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## Volumes of Solids

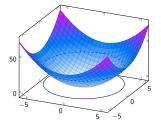
Find the volume under the paraboloid

$$z = x^2 + y^2$$

and above the disc

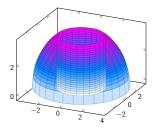
$$x^2 + y^2 < 25$$

- 1. Describe the disc in polar coordinates
- 2. Transform f(x, y) to polar coordinates



Integrals over Polar Rectangles

### Volumes of Solids



Find the volume inside the sphere

$$x^2 + y^2 + z^2 = 16$$

and outside the cylinder

$$x^2 + y^2 = 4$$

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