Math 213 - Vector-Valued Functions

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Homework

- Webwork A4 on section 12.6 is due Wednesday!
- Wednesday is Drop Day
- Quiz # 3 on 12.5–12.6 on Thursday
- Re-re-read section 13.1, pp. 848–853
- Begin working on pp. 853–855, problems 1-13 (odd), 17, 19, 21-26, 31, 43, 45, 49
- Read section 13.2, 855–859

Unit I: Geometry and Motion in Space

Lecture 1	Three-Dimensional Coordinate Systems
Lecture 2	Vectors
Lecture 3	The Dot Product
Lecture 4	The Cross Product
Lecture 5	Equations of Lines and Planes, Part I
Lecture 6	Equations of Lines and Planes, Part II
Lecture 7	Cylinders and Quadric Surfaces
Lecture 8	Vector Functions and Space Curves
Lecture 9	Derivatives and integrals of Vector Functions
Lecture 10	Motion in Space: Velocity and Acceleration
Lecture 11	Functions of Several Variables
Lecture 12	Exam 1 Review

Goals of the Day

- Understand what a **vector-valued function** is
- Understand limits and continuity for vector-valued functions
- Learn to visualize space curves:

  (i) by computing their projections onto the xy, xz, and yz planes,

  (ii) by viewing them as intersections of surfaces

Vector-Valued Functions

A **vector-valued function** is a function $\mathbf{r}(t)$ whose *domain* is a set of real numbers and whose *range* is a set of vectors in two- or three-dimensional space. We can specify $\mathbf{r}(t)$ through its *component functions*:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Example you already know: If $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{v} = \langle a, b, c \rangle$, then

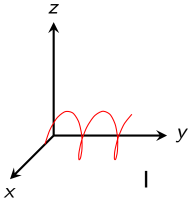
$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

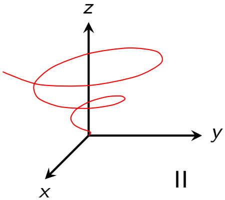
is a vector-valued function with component functions

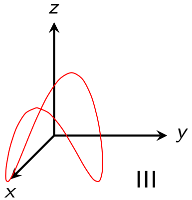
$$f(t) = x_0 + at, \quad g(t) = y_0 + bt, \quad h(t) = z_0 + ct$$

Vector Function Basics

1. What is the domain of the function $\mathbf{r}(t) = \left\langle \ln(t+1), \dfrac{t}{9-t^2}, 2^t \right\rangle$?
2. What is $\lim_{t\to 1} \left(\dfrac{t^2-t}{t-1}\mathbf{i} + \sqrt{t+8}\,\mathbf{j} + \dfrac{\sin \pi t}{\ln t}\mathbf{k} \right)$?
3. Can you match these curves with their graphs?
   (a) $\mathbf{r}(t) = \langle t\sin t, t\cos t, t\rangle$
   (b) $x = \cos t, y = \sin t, z = \cos 2t$
   (c) $x = \cos t, y = t, z = \sin t$







Breaking it Down: Limits and Continuity

The limit of a vector-valued function is the limit of the component functions:

$$\lim_{t \to t_0} \langle x(t), y(t), z(t) \rangle = \left\langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right\rangle$$

A vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is continuous at $t = a$ if each of the component functions $x(t)$, $y(t)$, $z(t)$ is continuous at $t = a$

In short, $\mathbf{r}(t)$ is continuous at a if $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$

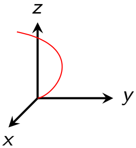
Space Curves

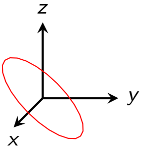
If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is a vector function defined on an interval I, the set of all points $(x(t), y(t), z(t))$ for t in the interval I is called a *space curve* C. The equations $x = f(t), y = g(t), z = h(t)$ are called the parametric equations for C.

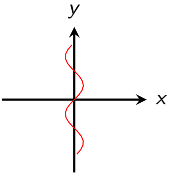
Match each of the space curves shown with their parametric equations.

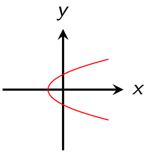
$\mathbf{r}(t) = \langle \sin t, t\rangle$ $\quad$ $\mathbf{r}(t) = \langle t^2 - 1, t\rangle$

$\mathbf{r}(t) = \langle t^2, t^3, t^4\rangle$ $\quad$ $\mathbf{r}(t) = \langle \cos t, -\cos t, \sin t\rangle$





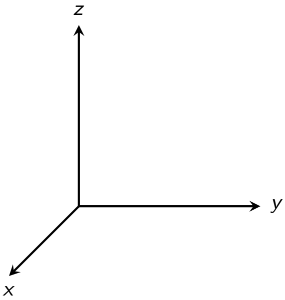




Line Segments are Space Curves

1. Find the vector equation and parametric equations for the line segment from $P(2, 0, 0)$ to $Q(6, 2, -2)$
2. Find the vector equation and parametric equations for the line segment from $P(a, b, c)$ to $Q(u, v, w)$

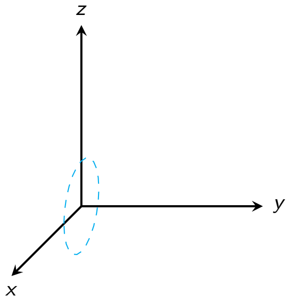
Visualizing It: Projections



Consider the space curve with parametric equations

$$x(t) = \cos t, \quad y(t) = t \quad z(t) = \sin t$$

Visualizing It: Projections

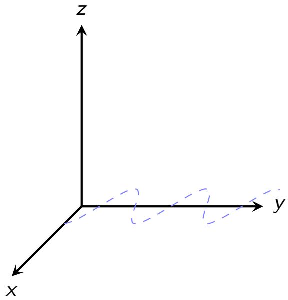


Consider the space curve with parametric equations

$$x(t) = \cos t, \quad y(t) = t \quad z(t) = \sin t$$

1. Find the projection of this curve onto the xz plane (the side wall)

Visualizing It: Projections



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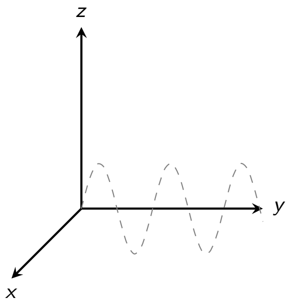
$$x(t) = \cos t, \quad y(t) = t \quad z(t) = \sin t$$

1. Find the projection of this curve onto the xz plane (the side wall)
2. Find the projection of this curve onto the xy plane (the floor)

Visualizing It: Projections



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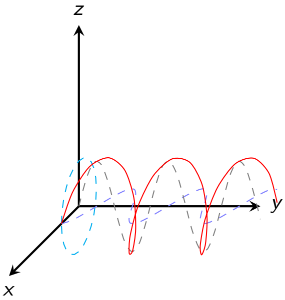
3. Find the projection of this curve onto the yz plane (the back wall)

Visualizing It: Projections



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1. Find the projection of this curve onto the *xz* plane (the side wall)
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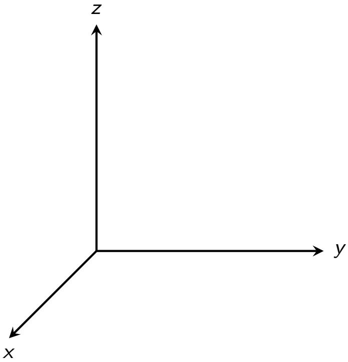
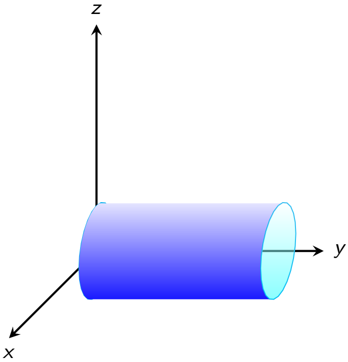
Visualizing It: Surfaces



Let's take another look at the curve

$$x(t) = \cos t, \quad y(t) = t \quad z(t) = \sin t$$

Visualizing It: Surfaces



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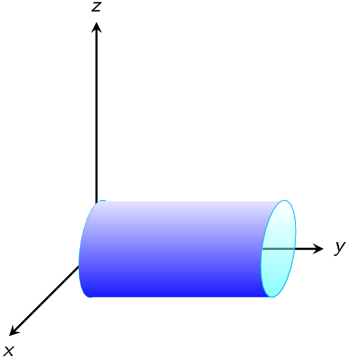
$$x(t) = \cos t, \quad y(t) = t \quad z(t) = \sin t$$

1. Show that this curve lies on the cylinder $x^2 + z^2 = 1$

Visualizing It: Surfaces



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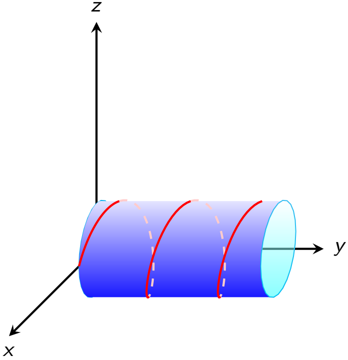
$$x(t) = \cos t, \quad y(t) = t \quad z(t) = \sin t$$

1. Show that this curve lies on the cylinder $x^2 + z^2 = 1$
2. Sketch the curve and the surface on the same set of axes

Visualizing It: Surfaces



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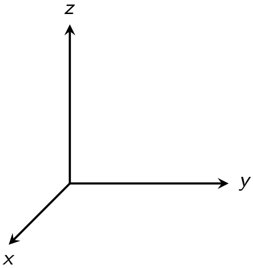
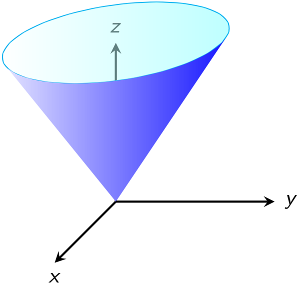


Let's revisit the curve

$$x(t) = t\cos t, \quad y(t) = t\sin t, \quad z(t) = t$$

1. Show that this curve lies on the right circular cone $z^2 = x^2 + y^2$
2. Sketch the curve and the surface on the same set of axes.

Visualizing It: Surfaces



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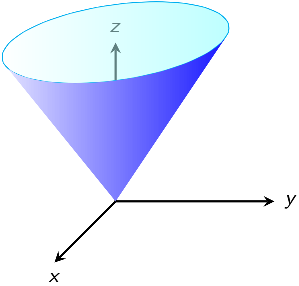

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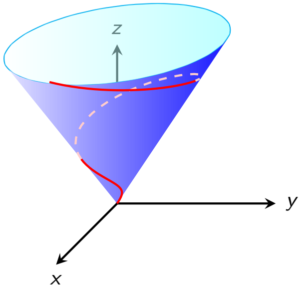

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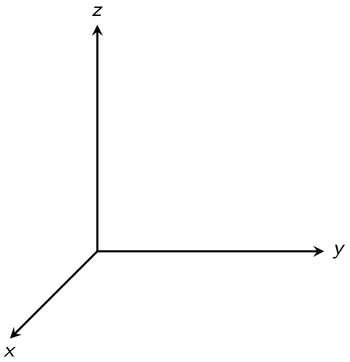
Visualize It: Space Curves and Surfaces

Show that the curve with parametric equations

$$x = \sin t, \quad y = \cos t, \quad z = \sin^2 t$$

is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$.



- The surface $z = x^2$ is a cylinder with curve $z = x^2$ parallel to the y-axis
- The surface $x^2 + y^2 = 1$ is a cylinder of radius 1 parallel to the the z-axis
- Their intersection is the parametric curve above

Visualize It: Space Curves and Surfaces

Show that the curve with parametric equations

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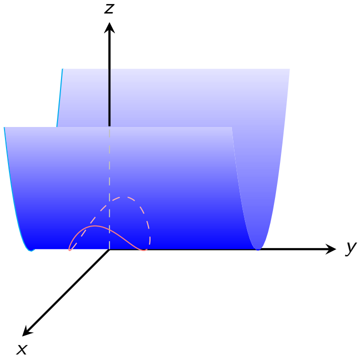

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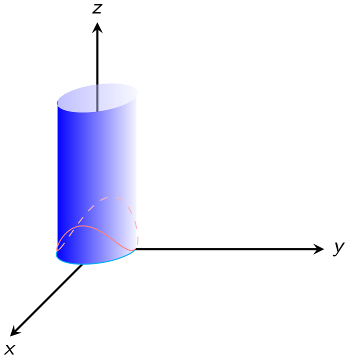

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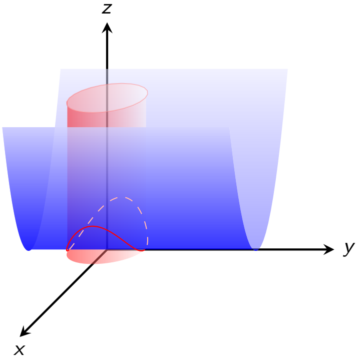

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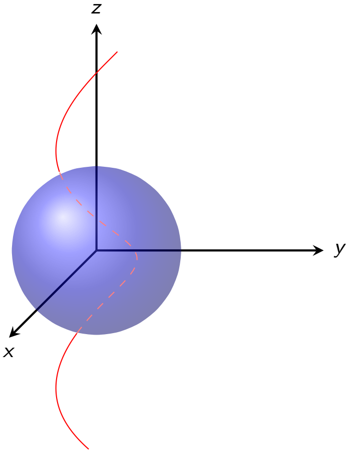
Intersections



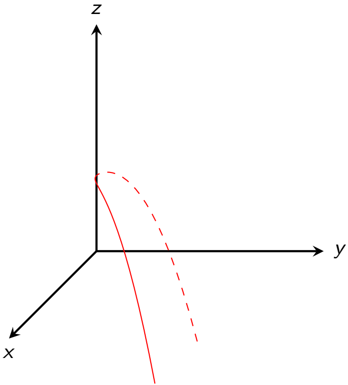
1. Find the points where the helix

$$\mathbf{r}(t) = \langle \sin(t), \cos(t), t \rangle$$

intersects the sphere

$$x^2 + y^2 + z^2 = 5$$

Intersections



1. Find the points where the helix
$$\mathbf{r}(t) = \langle \sin(t), \cos(t), t \rangle$$
intersects the sphere
$$x^2 + y^2 + z^2 = 5$$

2. Find the curve that describes the intersection of the parabolic cylinder $y = x^2$ and the top half of the ellipsoid
$$x^2 + 4y^2 + 4z^2 = 16$$

Intersections

Two particles travel along the space curves

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle, \quad \mathbf{r}_2(t) = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle$$

Do the particles collide? Do their paths intersect?

Recall:

Two particles *collide* if $\mathbf{r}_1(t) = \mathbf{r}_2(t)$ for the *same* t.

Two particles *intersect* if $\mathbf{r}_1(s) = \mathbf{r}_2(t)$ for (possibly different) times s and t.

Lecture Review

In this lecture:

- We learned what a vector-valued function
- We learned how to compute limits of space curves
- We learned what it means for a space curve to be continuous
- We learned how to visualize space curves as described by vector-valued functions