Exam 1

Name: ______________________ Section and/or TA: ____________

Last Four Digits of Student ID: ________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used except for a one-page sheet of formulas and facts. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 4 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. Unsupported answers on free response problems will receive *no credit*.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

10 (A) (B) (C) (D) (E)

SCORE

Multiple Choice	11	12	13	14	Total Score
50	10	10	15	15	100

Multiple Choice Questions

1. If $\mathbf{a} = \langle 1, -2, 1\rangle$ and $\mathbf{b} = \langle 3, 0, 2\rangle$ then $2\mathbf{a} + 3\mathbf{b} =$

   A. $\langle 11, -4, 8\rangle$

   B. $\langle 2, -4, 2\rangle$

   C. $\langle 9, 0, 4\rangle$

   D. $\langle 9, -6, 7\rangle$

   E. $\langle -9, 6, -7\rangle$

2. What is the distance of the point $(3, 4, 2)$ from the yz plane?

   A. 3

   B. 4

   C. 2

   D. 5

   E. $2\sqrt{5}$

3. Find the area of the triangle with vertices $Q(1, 0, 2)$, $R(2, 1, 3)$, $S(0, 1, 3)$.

   A. 1

   B. $2\sqrt{2}$

   C. $\sqrt{2}$

   D. $2\sqrt{5}$

   E. $\sqrt{5}$

4. Find the equation of a plane perpendicular to the vector $\mathbf{n} = \langle 2, -1, 3 \rangle$ and passing through the point $(-4, 2, 1)$.

   A. $x - 2y + 3z = -11$

   B. $2x - y + 3z = -2$

   C. $2x - y + 3z = 7$

   D. $2x - y + 3z = -7$

   E. $-x + 2y + 3z = 11$

5. Which of the following best describes the graph of the equation $z = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2$?

   A. Ellipse

   B. Elliptic cylinder

   C. Parabolic cylinder

   D. Elliptic paraboloid

   E. Hyperbolic paraboloid

6. The function $\mathbf{r}(t) = 4\cos(t)\mathbf{i} + \mathbf{j} + 4\sin(t)\mathbf{k}$ traces out:

   A. A circle of radius 4 and center $(0, 1, 0)$ in the plane $y = 1$

   B. A circle of radius 4 and center $(0, 1, 0)$ in the plane $x + y = 8$

   C. A circle of radius 4 and center $(1, 0, 0)$ in the plane $x = 1$

   D. A circle of radius 2 and center $(0, 1, 0)$ in the plane $x = 1$

   E. A circle of radius 2 and center $(0, 1, 0)$ in the plane $y = 1$

7. Which of the following integrals correctly computes the arc length of the curve $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ between $t = 0$ and $t = 1$?

   A. $\int_0^1 \sqrt{t^2 + t^4 + t^6}\, dt$

   B. $\int_0^1 \sqrt{1 + 4t^2 + 9t^4}\, dt$

   C. $\int_0^1 \sqrt{1 + 2t^2 + 3t^6}\, dt$

   D. $\int_0^1 \sqrt{t + t^2 + t^3}\, dt$

   E. $\int_0^1 \sqrt{1 + 2t + 3t^2}\, dt$

8. Find the velocity and speed if $\mathbf{r}(t) = \langle \cos(3t), \sin(3t), 4t \rangle$

   A. $\mathbf{r}'(t) = \langle \frac{1}{3}\sin(3t), -\frac{1}{3}\cos(3t), 2t^2 \rangle$ and $|\mathbf{r}'(t)| = \sqrt{4t^4 + 1/9}$

   B. $\mathbf{r}'(t) = \langle -\frac{1}{3}\sin(3t), \frac{1}{3}\cos(3t), 2t^2 \rangle$ and $|\mathbf{r}'(t)| = \sqrt{4t^4 + 1/9}$

   C. $\mathbf{r}'(t) = \langle 3\sin(3t), -\cos(3t), 4 \rangle$ and $|\mathbf{r}'(t)| = \sqrt{5}$

   D. $\mathbf{r}'(t) = \langle -3\sin(3t), \cos(3t), 4 \rangle$ and $|\mathbf{r}'(t)| = \sqrt{5}$

   E. $\mathbf{r}'(t) = \langle -3\sin(3t), 3\cos(3t), 4 \rangle$ **and** $|\mathbf{r}'(t)| = 5$

9. Consider the planes given by the equations

$$x + 2y - z = 2$$
$$2x - 2y + z = 1$$

   Which one of the following statements is correct?

   A. These planes are parallel

   B. These planes are skew

   C. These planes intersect one another and the vector $\mathbf{v} = \langle 0, 3, 6 \rangle$ points along the line of intersection

   D. These planes intersect one another and the vector $\mathbf{v} = \langle 1, 2, -1 \rangle$ points along the line of intersection

   E. These planes intersect one another and the vector $\mathbf{v} = \langle 2, -2, 1 \rangle$ points along the line of intersection

10. Which of the following is *not* a well-defined operation on vectors?

   A. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

   B. $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$

   C. $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{c})$

   D. $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

   E. $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Free Response Questions

11. (10 points) Find the volume of a parallelepiped with adjacent edges PQ, PR, and PS if $P = (3, 0, 1)$, $Q = (-1, 2, 5)$, $R = (5, 1, -1)$, $S = (0, 4, 2)$

Solution: Compute

$$\overrightarrow{PQ} = \langle -4, 2, 4 \rangle \qquad \text{(2 points)}$$
$$\overrightarrow{PR} = \langle 2, 1, -2 \rangle \qquad \text{(2 points)}$$
$$\overrightarrow{PS} = \langle -3, 4, 1 \rangle \qquad \text{(2 points)}$$

The signed volume of the parallelepiped is

$$\overrightarrow{PQ} \cdot \left(\overrightarrow{PR} \times \overrightarrow{PS} \right) = \begin{vmatrix} -4 & 2 & 4 \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix} \qquad \text{(2 points)}$$
$$= -16 \qquad \text{(2 points)}$$

The unsigned volume is 16.

12. (10 points) The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2 = \langle \sin t, \sin 2t, t \rangle$ intersect at $t = 0$. Find the angle of intersection to the nearest degree.

Solution:

$$\mathbf{r}_1'(t) = \langle 1, 2t, 3t^2 \rangle$$
$$\mathbf{r}_1'(0) = \langle 1, 0, 0 \rangle \qquad \text{(2 points)}$$

$$\mathbf{r}_2'(t) = \langle \cos t, 2\cos 2t, 1 \rangle$$
$$\mathbf{r}_2'(0) = \langle 1, 2, 1 \rangle \qquad \text{(2 points)}$$

Hence

$$\cos(\theta) = \frac{\mathbf{r}_1'(0) \cdot \mathbf{r}_2'(0)}{|\mathbf{r}_1'(0)|\,|\mathbf{r}_2'(0)|} \qquad \text{(2 points)}$$
$$= \frac{1}{\sqrt{6}} \qquad \text{(1 points)}$$

so that

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) = 66^\circ \qquad \text{(1 points)}$$

One common problem was that some students did not remember to take derivatives to find the tangent. A bonus of 2 points was awarded for attempts (successful or otherwise) to take derivatives.

13. (15 points) (a) (7 points) Find a vector function $\mathbf{r}(t)$ that represents the intersection of the hyperboloid $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$.

Solution: There are several correct solutions–here is one.

We can parameterize motion on the circle at

$$x(t) = \cos(t),\ y(t) = \sin(t) \quad \text{(2 points)}$$

Next, we use the equation of the hyperboloid to conclude that

$$z = \cos^2(t) - \sin^2(t) \quad \text{(2 points)}$$

Putting all the pieces together, we conclude that

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), \cos^2(t) - \sin^2(t) \rangle. \quad \text{(2 points)}$$

A bonus of one point was awarded for any attempt to substitute into the equations of the hyperboloid and the cylinder.

(b) (8 points) Two objects travel through space with trajectories given by the vector functions

$$\mathbf{r}_1(t) = \langle t^2, 7t - 12, t^2 \rangle, \quad \mathbf{r}_2(t) = \langle 4t - 3, t^2, 5t - 6 \rangle.$$

Determine whether these objects collide and, if so, find the coordinates of the collision point.

Solution: We seek a value of t which makes the x-, y- and z-components equal. Equating x components we get

$$t^2 - 4t + 3 = 0$$

so that $(t-3)(t-1) = 0$. (2 points)

To check for an intersection, we try each of the values $t = 1$ and $t = 3$:

t	$\mathbf{r}_1(t)$	$\mathbf{r}_2(t)$
1	$\langle 1, -5, 1 \rangle$	$\langle 1, 1, -1 \rangle$ (2 points)
3	$\langle 9, 9, 9 \rangle$	$\langle 9, 9, 9 \rangle$ (2 points)

(Remark: students were not penalized if they did not test $t = 1$)

We conclude that the two particles collide at

$$t = 3 \quad \text{(1 points)}$$

and coordinates

$$\langle 9, 9, 9 \rangle. \quad \text{(1 points)}$$

14. (15 points) A projectile is fired with an initial speed of 800 ft/sec at an angle of elevation of 45°, so that its initial velocity is

$$\mathbf{v}(0) = 400\sqrt{2}\mathbf{i} + 400\sqrt{2}\mathbf{k}.$$

(a) (3 points) Find the velocity of the projectile as a function of time, given that the acceleration vector is the constant vector $\mathbf{a}(t) = -32\mathbf{k}$.

Solution: $\mathbf{v}(t) = 400\sqrt{2}\mathbf{i} + (400\sqrt{2} - 32t)\mathbf{k}$ (3 points)

(b) (4 points) Find the position of the projectile as a function of time.

Solution:

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}(0) + \int_0^t \mathbf{v}(s)\, ds && \text{(2 points)} \\ &= ([400\sqrt{2}]t)\mathbf{i} + ([400\sqrt{2}]t - 16t^2)\mathbf{k} && \text{(2 points)} \end{aligned}$$

(c) (4 points) At what time does the projectile hit the ground?

Solution: The projectile hits the ground with the $\mathbf{k}$-component of $\mathbf{r}(t)$ is zero, so

$$[400\sqrt{2}]t - 16t^2 = 0 \quad \text{(2 points)}$$

Solving for t we get

$$t = 25\sqrt{2}. \quad \text{(2 points)}$$

(d) (4 points) How far away from the starting point does the projectile hit the ground?

Solution: Find the $\mathbf{i}$-component of $\mathbf{r}(t)$ at the time found in part (c).

$$\begin{aligned} x(25\sqrt{2}) &= [400\sqrt{2}]25\sqrt{2} && \text{(2 points)} \\ &= 20,000 \text{ ft} && \text{(2 points)} \end{aligned}$$