Math 213 Exam 1

Name: ____________________ Section: __________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used other than a one-page "cheat sheet" of notes, formulas, etc., written or typeset on one or both sides of an 8-1/2" $\times$ 11" sheet of paper. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 6 free-response questions. Please follow these guideines to receive maximum credit.

- Each question is followed by a space to write your answer. Please write your answer *neatly* in the space provided.
- Show all work to receive full credit on the free response problems. You will be graded on the clarity of your presentation as well as the correctness of your answers.
- Give exact answers, rather than decimal equivalents, unless otherwise instructed (e.g., $\sqrt{2}$, not 1.414).

Question	1	2	3	4	5	6	**Total**
Possible	10	18	18	18	18	18	100
Score							

1. (Vector Basics - 10 points) Suppose that $\mathbf{a} = \langle 3, 4\rangle$, $\mathbf{b} = \langle 9, -1\rangle$. Find:

(a) (3 points) $4\mathbf{a} + 2\mathbf{b}$

Solution:

$$\begin{aligned} 4\mathbf{a} + 2\mathbf{b} &= 4\langle 3, 4\rangle + 2\langle 9, -1\rangle && \text{(1 points)} \\ &= \langle 12, 16\rangle + \langle 18, -2\rangle && \text{(1 points)} \\ &= \langle 30, 14\rangle && \text{(1 points)} \end{aligned}$$

(b) (4 points) $|\mathbf{a} - \mathbf{b}|$

Solution:

$$\begin{aligned} \mathbf{a} - \mathbf{b} &= \langle 3 - 9, 4 + 1\rangle && \text{(1 points)} \\ &= \langle -6, 5\rangle && \text{(1 points)} \\ |\mathbf{a} - \mathbf{b}| &= \sqrt{5^2 + 6^2} = \sqrt{61} && \text{(2 points)} \end{aligned}$$

(c) (3 points) $\mathbf{a} \cdot \mathbf{b}$

Solution:

$$\mathbf{a} \cdot \mathbf{b} = 27 - 4 = 23 \qquad \text{(3 points)}$$

2. (Dot and Cross Products - 18 points)

(a) (4 points) If $\mathbf{a} = \langle 2, -1, 3 \rangle$ and $\mathbf{b} = \langle 4, 2, 1 \rangle$, find $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$.

Solution:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 4 & 2 & 1 \end{vmatrix} && \text{(1 points)} \\ &= -7\mathbf{i} + 10\mathbf{j} + 8\mathbf{k} && \text{(1 points)} \\ \mathbf{b} \times \mathbf{a} &= -\mathbf{a} \times \mathbf{b} && \text{(1 points)} \\ &= 7\mathbf{i} - 10\mathbf{j} - 8\mathbf{k} && \text{(1 points)} \end{aligned}$$

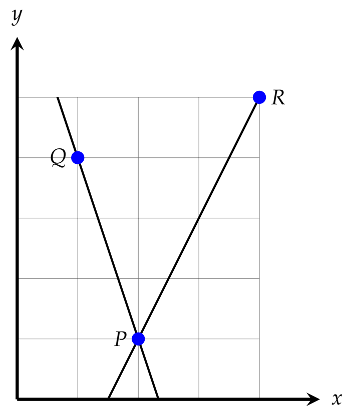
(b) (4 points) Use the scalar triple product to verify that the vectors $\mathbf{u} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$, and $\mathbf{w} = 5\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}$ are coplanar. Be sure to justify your conclusion.

Solution: Three vectors are coplanar if their scalar triple product is zero

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix} && \text{(1 points)} \\ &= 1 \begin{vmatrix} -1 & 0 \\ 9 & -4 \end{vmatrix} - 5 \begin{vmatrix} 3 & 0 \\ 5 & -4 \end{vmatrix} + (-2) \begin{vmatrix} 3 & -1 \\ 5 & 9 \end{vmatrix} && \text{(1 points)} \\ &= 0 && \text{(1 points)} \end{aligned}$$

which shows that these vectors are coplanar. (1 points)

(c) (10 points) Using dot products, find the acute angle between the lines $2x - y = 3$ and $3x + y = 7$ shown below. Be sure to explain how the dot product is used, with what vectors, and why!



Solution: Use vectors along line (these can be read off from the equations)

$$\mathbf{v}_1 = \langle 2, -1 \rangle \qquad \text{(1 points)}$$
$$\mathbf{v}_2 = \langle 3, 1 \rangle \qquad \text{(1 points)}$$
$$|\mathbf{v}_1| = \sqrt{5} \qquad \text{(1 points)}$$
$$|\mathbf{v}_2| = \sqrt{10} \qquad \text{(1 points)}$$
$$\cos\theta = \frac{5}{\sqrt{5}\sqrt{10}} = \frac{1}{\sqrt{2}} \qquad \text{(4 points)}$$
$$\theta = \frac{\pi}{4} \text{ radians} \qquad \text{(2 points)}$$

Alternatively one can pick points along the lines:

$$\overrightarrow{PQ} = \langle -1, 3 \rangle, \quad \left|\overrightarrow{PQ}\right| = \sqrt{10} \qquad \text{(2 points)}$$
$$\overrightarrow{PR} = \langle 2, 4 \rangle \quad \left|\overrightarrow{PR}\right| = \sqrt{20} \qquad \text{(2 points)}$$
$$\overrightarrow{PQ} \cdot \overrightarrow{PR} = -2 + 12 = 10 \qquad \text{(2 points)}$$
$$\cos\theta = \frac{10}{\sqrt{10} \cdot \sqrt{20}} = \frac{1}{\sqrt{2}} \qquad \text{(2 points)}$$
$$\theta = \frac{\pi}{4} \text{ radians} \qquad \text{(2 points)}$$

3. (18 points - Equations of Lines and Planes)

(a) (6 points) Find the parametric and symmetric equations for the line through the points $(0,0,0)$ and $(4,3,1)$.

Solution: We use $(0,0,0)$ as a point on the line and $\langle 4,3,1\rangle$ as a vector pointing along the line. Parametric equations:

$$x(t) = 4t, \quad y(t) = 3t, \quad z(t) = t \quad \text{((3 points))}$$

Symmetric equations:

$$\frac{x}{4} = \frac{y}{3} = \frac{z}{1} \quad \text{(3 points)}$$

(b) (6 points) Find the equation of the plane through the origin and perpendicular to the vector $\langle 1,-2,4\rangle$.

Solution: We use $(x,y,z) = (0,0,0)$ as a point on the plane and $\langle 1,-2,4\rangle$ as a vector perpendicular to the plane ((2 points))
The equation is

$$x - 2y + 4z = 0 \quad \text{(4 points)}$$

(c) (6 points) Determine whether the planes $x + 2y - z = 2$ and $2x - 2y + z = 1$ are parallel, perpendicular, or neither.

Solution: Normal vectors for the two planes are

$$\mathbf{n}_1 = \langle 1,2,-1\rangle, \quad \mathbf{n}_2 = \langle 2,-2,1\rangle \quad \text{(1 points)}$$

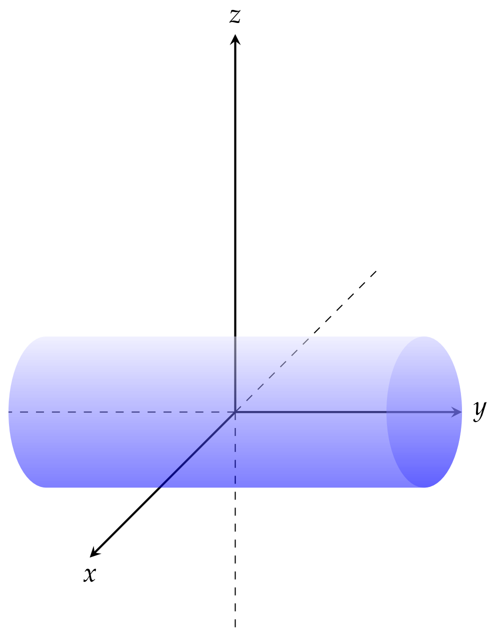
We compute

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & -2 & 1 \end{vmatrix} = -4\mathbf{i} + 3\mathbf{j} - 6\mathbf{k} \neq 0$$

so the normals, and hence the planes, are *not* parallel (2 points) *Remark*: Students can also observe that $\mathbf{n}_1$ is not a multiple of $\mathbf{n}_2$, provided some justification is given
We compute $\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 4 - 1 \neq 0$ so the normals, and hence the planes, are not perpendicular. (2 points)
Thus these planes are neither parallel nor perpendicular. (1 points)

4. (Quadric Surfaces - 18 points)

(a) (9 points) Sketch the surface $x^2 + z^2 = 1$ on the axes provided. Also, please provide the information requested below the sketch.



Solution: Scoring for the sketch:

Cylindrical shape (1 points)
Symmetry axis is y axis (1 points)
Overall quality (1 points)

Specify the axis of symmetry:

Solution: The axis of symmetry is the y axis (2 points)

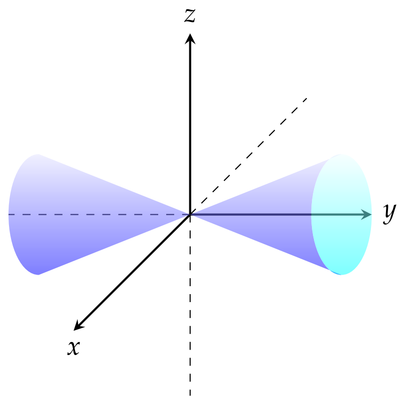
Describe the cross-sections perpendicular to the axis of symmetry.

Solution: The cross sections are circles (1 points) of radius 1 (1 points) with center on the y-axis (1 points)

Name the surface:

Solution: Right circular cylinder (1 points)

(b) (9 points) Sketch the surface $x^2 + z^2 = y^2$ on the axes provided. Also, please provide the information requested below your sketch.



Scoring for graph:

Cones on correct axis (1 points)
Conical shape (1 points)

Identify the traces in planes parallel to the xy-plane by equation and by name.

Solution: $x^2 + k^2 = y^2$ or $y^2 - x^2 = k^2$ (hyperbola, opening along y-axis, or lines $y = \pm x$ if $k = 0$)
(2 points)

Identify the traces in planes parallel to the xz-plane by equation and by name.

Solution: $x^2 + z^2 = k^2$ (circles of radius k with centers on the y-axis)
(2 points)

Identify the traces in planes parallel to the yz-plane by equation and by name.

Solution: $k^2 + z^2 = y^2$ or $y^2 - z^2 = k^2$ (hyperbola opening along y-axis)
(2 points)

Name the surface:

Solution: Cone (1 points)

5. (Vector Functions, Derivatives, Integrals - 18 points)

(a) (4 points) Find the unit tangent $\mathbf{T}(t)$ to the curve

$$\mathbf{r}(t) = \langle t^2 - 2t, 1 + 3t, t^3/3 + t^2/2 \rangle$$

at $t = 2$.

Solution:

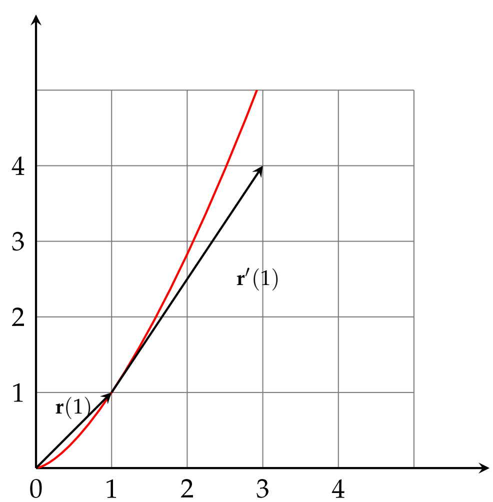
$$\mathbf{r}'(t) = \langle 2t - 2, 3, t^2 + t \rangle \quad \text{(1 points)}$$
$$\mathbf{r}'(2) = \langle 2, 3, 6 \rangle \quad \text{(1 points)}$$
$$|\mathbf{r}'(2)| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7 \quad \text{(1 points)}$$
$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{|\mathbf{r}'(2)|} = \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle \quad \text{(1 points)}$$

(b) (3 points) Find a formula for $\mathbf{r}'(t)$ for the vector function $\mathbf{r}(t) = \langle t^2, t^3 \rangle$

Solution:

$$\mathbf{r}'(t) = \langle 2t, 3t^2 \rangle \quad \text{(3 points)}$$

(c) (4 points) Below is the curve $\mathbf{r}(t)$ from part (b) for $0 \leq t \leq \sqrt{3}$. Sketch and label the position vector $\mathbf{r}(1)$ starting at the origin and label the point P having position vector $\mathbf{r}(1)$. Then sketch and label the tangent vector $\mathbf{r}'(1)$ starting at the point P.



$$\mathbf{r}(1) = \langle 1, 1 \rangle \quad \text{(1 points)}$$
$$\mathbf{r}'(1) = \langle 2, 3 \rangle \quad \text{(1 points)}$$

Sketch of $\mathbf{r}$, $\mathbf{r}'$ as shown: (2 points)

(d) (7 points) Find a vector function that represents the intersection of the hyperbolic paraboloid ("saddle") $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$. *Hint*: First parameterize x and y to move along the circle of radius 1.

Solution: Following the hint, set

$$x(t) = \cos(t) \qquad \text{(1 points)}$$
$$y(t) = \sin(t) \qquad \text{(1 points)}$$

with $z(t)$ to be determined. Note that $(x(t), y(t), z(t)$ lies on the cylinder for *any* function $z(t)$. (1 points)

So that the curve moves on the saddle, solve for $z(t)$:

$$z(t) = x(t)^2 - y(t)^2 \qquad \text{(1 points)}$$
$$z(t) = \cos^2(t) - \sin^2(t) \qquad \text{(1 points)}$$
$$\mathbf{r}(t) = \left\langle \cos(t), \sin(t), \cos^2(t) - \sin^2(t) \right\rangle \qquad \text{(2 points)}$$

Remark: There are other correct answers. One could also set $x(t) = \sin(t)$, $y(t) = \cos(t)$ and get $z(t) = \sin^2(t) - \cos^2(t)$

6. (Motion in Space - 18 points)

A projectile is fired with initial speed of 200 m/s and angle of elevation 60°. Recall that the acceleration due to gravity is 9.8 m/sec^2.

(a) (3 points) Find the initial velocity vector $\mathbf{v}_0$.

Solution:

$$\begin{aligned} \mathbf{v}(0) &= \langle 200\cos 60^\circ, 200\sin 60^\circ\rangle & \text{(1 points)} \\ &= \langle 100, 100\sqrt{3}\rangle & \text{(2 points)} \end{aligned}$$

(b) (3 points) Find the equation for $\mathbf{r}(t)$, the position of the projectile as a function of time.

Solution:

$$\begin{aligned} \mathbf{a}(t) &= \langle 0, -9.8\rangle & \text{(1 points)} \\ \mathbf{v}(t) &= \mathbf{v}(0) + \langle 0, -9.8t\rangle = \langle 100, 100\sqrt{3} - 9.8t\rangle & \text{(1 points)} \\ \mathbf{r}(t) &= \langle 100t, 100\sqrt{3}t - 4.9t^2\rangle & \text{(1 points)} \end{aligned}$$

If students don't integrate but instead use a prepackaged, cheat-sheeted formula, this is OK so long as they state the formula in general and explain how it's applied to the situation at hand.

(c) (3 points) Find the time of impact.

Solution: Impact occurs when $y(t) = 0$ (1 points)
Solve

$$\begin{aligned} 100\sqrt{3}t - 4.9t^2 &= 0 \\ t\left(100\sqrt{3} - 4.9t\right) &= 0 & \text{(1 points)} \\ t_i &= \frac{100\sqrt{3}}{4.9} \simeq 35.34 \text{ sec} & \text{(1 points)} \end{aligned}$$

(d) (3 points) Find the maximum height reached.

Solution: To find the time of maximum height, either (i) explain that $t = t_i/2$

(1 points) or (ii) solve the the time when the y-velocity is zero:

$$\begin{aligned} 100\sqrt{3} - 9.8t &= 0 \\ t &= \frac{100\sqrt{3}}{9.8} \simeq 17.67 \text{ sec} \qquad \text{(1 points)} \end{aligned}$$

Now use this value of t in the equation for $y(t)$ (1 points)

$$\begin{aligned} y(17.67) &= 100\sqrt{3}(17.67) - 4.9(17.67)^2 \\ &= 1530.61 \text{ m} \qquad \text{(1 points)} \end{aligned}$$

(e) (3 points) Find an expression for $\mathbf{r}'(t)$, the velocity of the projectile.

Solution:

$$\mathbf{r}'(t) = \left\langle 100, 100\sqrt{3} - 9.8t \right\rangle \qquad \text{(3 points)}$$

If students didn't get this on the way to solving part (a), they should indicate (implicitly or explicitly) that they took the derivative of $\mathbf{r}(t)$ from a previous step

(f) (3 points) Find the speed of the projectile at impact.

Solution: I'll leave the scoring of this part to the grading group!
The likely student solution will be numeric and look something like this.

$$\begin{aligned} \mathbf{r}'(t_i) &= \left\langle 100, 100\sqrt{3} - 9.8 * (35.34) \right\rangle \\ &= \langle 100, -173.13 \rangle \end{aligned}$$

and then compute the magnitude. If, however, the students either keep exact expressions throughout or realize that, by symmetry, the velocty vector at impact should be the mirror image of the velocity vector at launch, they'll get

$$\mathbf{r}'(t_i) = \left\langle 100, -100\sqrt{3} \right\rangle$$

and so

$$|\mathbf{r}'(t_i)| = \sqrt{100^2 + 100^2 \cdot 3} = 200 \text{ m /sec}$$