**MA 213 — Calculus III** Spring 2018
**Exam 1** February 8, 2018

**Name:** KEY

**Section:** ____________

**Last 4 digits of student ID #:** ________

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- All questions are free response questions. Show all your work on the page of the problem. Clearly indicate your answer and the reasoning used to arrive at that answer. Unsupported answers may not receive credit.

# Exam Scores

*Do not write in the table below*

| Question | Score | Total |
|---|---|---|
| 1 | | 10 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| Total | | 100 |

**Free Response. Show your work!**

1. (10 points) At what points does the curve $\mathbf{r}(t) = t\mathbf{i} + (4t - t^2)\mathbf{k}$ intersect the paraboloid $z = x^2 + y^2$?

Find $t$ so that $z(t) = x(t)^2 + y(t)^2$

$\Leftrightarrow \quad 4t - t^2 = t^2 \quad$ (as $y(t) \equiv 0$)

$4t - 2t^2 = 0$

$2t(2-t) = 0 \Rightarrow t = 0, t = 2$

The points are $\boxed{r(0) = \langle 0,0,0 \rangle, \quad r(2) = (2,0,4)}$

2. (10 points) A curve $C$ is represented by the vector function

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + 3t\mathbf{j} + 2\sin(2t)\mathbf{k}.$$

Find the unit tangent vector to $C$ at the point where $t = 0$.

$\vec{r}'(t) = -\sin(t)\hat{i} + 3\hat{j} + 4\cos(2t)\hat{k}$

$\vec{r}'(0) = 0\hat{i} + 3\hat{j} + 4\hat{k}$

$|\vec{r}'(0)| = \sqrt{3^2+4^2} = 5$

$\vec{T}(0) = \dfrac{\vec{r}'(0)}{|\vec{r}'(0)|} = \dfrac{1}{5}(3\hat{j} + 4\hat{k}) = \boxed{\dfrac{3}{5}\hat{j} + \dfrac{4}{5}\hat{k}}$

## Free Response. Show your work!

**3.** (10 points) Evaluate the limit

$$\lim_{t\to 0}\left(e^t\,\mathbf{i}+\left(\frac{\sin 2t}{t}\right)\mathbf{j}+(\tan t)\mathbf{k}\right).$$

① $\lim_{t\to 0} e^t = 1$

② $\lim_{t\to 0} \frac{\sin(2t)}{t} = \lim_{t\to 0} \frac{2\cos 2t}{1} = 2$

③ $\lim_{t\to 0} \tan t = \lim_{t\to 0} \frac{\sin t}{\cos t} = 0$

$\therefore \lim_{t\to 0}\left(e^t\hat{i} + \left(\frac{\sin 2t}{t}\right)\hat{j} + (\tan t)\hat{k}\right) = \hat{i} + 2\hat{j}$

**4.** (10 points) Identify the surface $9y^2 - 4z^2 = x^2 + 36$ as one of the following types:

a. Cylinder

b. Ellipsoid

c. Elliptic Paraboloid

d. Hyperbolic Paraboloid

e. Cone

f. Hyperboloid of One Sheet

g. Hyperboloid of Two Sheets

In standard form: $9y^2 - 4z^2 - x^2 = 36$

$\div 36$ $\quad \frac{y^2}{4} - \frac{z^2}{9} - \frac{x^2}{36} = 1$

Because signs are $(+ - -)$ this is the equation of a hyperboloid of two sheets

**Free Response. Show your work!**

5. (10 points) Which of the following four planes are parallel? Are any of them identical?

$$P_1 : 3x + 6y - 3z = 6 \qquad P_2 : \; 4x - 12y + 8z = 5$$
$$P_3 : 9y = 1 + 3x + 6z \qquad P_4 : \; z = x + 2y - 2$$

Normal vectors: $n_1 = \langle 3, 6, -3 \rangle$ $\quad n_2 = \langle 4, -12, 8 \rangle$

$n_3 = \langle 3, -9, 6 \rangle$ $\quad n_4 = \langle 1, 2, -1 \rangle$

Notice $n_1 = 3n_4$ so $\boxed{P_1 \parallel P_4}$ No other parallel planes.

Take eq'n for $P_1$: $3x + 6y - 3z = 6$

$\div 3$ $\quad x + 2y - z = 2$

Solve for $z$: $\quad z = x + 2y - 2$

Hence $\boxed{P_4 = P_1}$

6. (10 points) Determine whether the lines $L_1$ and $L_2$ intersect, and if they do, find the point of intersection.

$$L_1 : \quad x = -1 + 3t, \qquad y = 3 - t, \qquad z = -3 + 2t$$
$$L_2 : \quad x = -3 - 5s, \qquad y = 4 + 2s, \qquad z = -4 - 3s.$$

(1) $-1 + 3t = -3 - 5s$

(2) $3 - t = 4 + 2s$

(3) $-3 + 2t = -4 - 3s$

From (2) $\quad t = -1 - 2s$

Substitute into (1): $-1 + 3(-1-2s) = -3 - 5s$

$\therefore -4 - 6s = -3 - 5s$

$-1 = s$

$\therefore t = -1 - 2(-1) = 1$

Check $s = -1$, $t = 1$

(1) $-1 + 3 = -3 + 5$ ✓

(2) $3 - 1 = 4 + 2(-1)$ ✓

(3) $-3 + 2(1) = -4 - 3(-1)$ ✓

Intersection at

$$\boxed{\begin{aligned} x &= -1 + 3 = 2 \\ y &= 3 - 1 = 2 \\ z &= -3 + 2 = -1 \end{aligned}}$$

**Free Response. Show your work!**

7. (10 points) Find an equation for the plane through the points $\overset{P}{(0,1,2)}, \overset{Q}{(1,0,2)}$, and $\underset{R}{(1,2,0)}$. Write the equation of the plane in the form $\underline{2x + by + z = d}$. Not possible!

Find a normal vector: $\vec{PQ} = \langle 1,-1,0\rangle \quad \vec{PR} = \langle 1,1,-2\rangle$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ +1 & -1 & 0 \\ 1 & 1 & -2 \end{vmatrix} = +2\hat{i} + +2\hat{j} + 2\hat{k}$$

$\therefore\ 2x+2y+2z=d \quad$ using $(0,1,2)$, $d=6$

$$\boxed{2x+2y+2z=6}$$

8. (10 points) Find the volume of the parallelepiped with adjacent edges $PQ, PR$, and $PS$, where

$$P = (-2,1,0), \quad Q = (2,3,2), \quad R = (1,4,-1), \quad S = (3,6,1).$$

$\vec{PQ} = \langle 4,2,2\rangle \quad \vec{PR} = \langle 3,3,-1\rangle \quad \vec{PS} = \langle 5,5,1\rangle$

$$V = |\vec{PQ}\cdot(\vec{PR}\times\vec{PS})|$$

$$= \begin{vmatrix} 4 & 2 & 2 \\ 3 & 3 & -1 \\ 5 & 5 & 1 \end{vmatrix} = 4\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 3 \\ 5 & 5 \end{vmatrix}$$

$$= 4(3+5) - 2(3+5) + 2\cdot 0$$

$$= \boxed{16}$$

**Free Response. Show your work!**

**9.** (10 points) Find the acute angle between the lines $3x - y = 7$ and $2x + y = 3$. [An exact answer in radians is expected. Approximate answers will nor receive full credit.]

$L_1$ $y = 3x - 7$ so $r(t) = \langle t, 3t-7 \rangle$ vector $u_1 = \langle 1, 3 \rangle$ along $L_1$

$L_2$ : $y = 3 - 2x$ so $r(t) = \langle t, 3 - 2t \rangle$ vector $u_2 = \langle 1, -2 \rangle$ along $L_2$

$$\cos\theta = \frac{u_1 \cdot u_2}{|u_1|\,|u_2|} = \frac{\langle 1,3\rangle \cdot \langle 1,-2\rangle}{\sqrt{1^2+3^2}\cdot\sqrt{1^2+2^2}} = \frac{-5}{\sqrt{10}\cdot\sqrt{5}} = -\sqrt{\frac{5}{10}} = -\sqrt{\frac{1}{2}}$$

$$\cos\theta = -\frac{1}{\sqrt{2}} \qquad \boxed{\theta = -\frac{\pi}{4}}$$

**10.** (10 points) Find the work done by a force $\mathbf{F} = 8\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$ that moves an object from the point $(0, 10, 8)$ to the point $(6, 12, 20)$. The distance is measured in meters and the force in newtons.

$\vec{r} = \langle 6, 2, 12 \rangle$ (displacement) from $(0,10,8)$ to $(6,12,20)$

$$\vec{F} \cdot \vec{r} = \langle 8, -6, 9 \rangle \cdot \langle 6, 2, 12 \rangle$$

$$= 48 - 12 + 108$$

$$= \boxed{144 \text{ nt} \cdot \text{m}}$$

$$\begin{array}{r} 108 \\ 48 \\ \hline 156 \\ -12 \\ \hline 144 \end{array}$$