MA 213 — Calculus III Spring 2018
Exam 2 March 8, 2018

Name: KEY

Section: __________

Last 4 digits of student ID #: ______

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- All questions are free response questions. Show all your work on the page of the problem. Clearly indicate your answer and the reasoning used to arrive at that answer. Unsupported answers may not receive credit.

Exam Scores

Do not write in the table below

Question	Score	Total
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Free Response. Show your work!

1. (10 points) Find the length of the curve

$$\mathbf{r}(t) = \sqrt{2}\,t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}, \quad (0 \le t \le 1).$$

Exact answer is expected. [Hint: Factor the expression under the square root.]

$\vec{r}'(t) = \sqrt{2}\,\hat{i} + e^t\hat{j} - e^{-t}\hat{k}$

$|\vec{r}'(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$

$s = \int_0^1 |\vec{r}'(t)|\,dt = \int_0^1 (e^t + e^{-t})\,dt = [e^t - e^{-t}]\Big|_0^1 = e - e^{-1}$

2. (10 points) Compute the curvature of the curve

$$\mathbf{r}(t) = t^3\mathbf{j} + t^2\mathbf{k}$$

at the point $(0, 1, 1)$.

Recall $\kappa(t) = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^2}$

$\vec{r}'(t) = 3t^2\hat{j} + 2t\hat{k}$

$\vec{r}''(t) = 6t\hat{j} + 2\hat{k}$

$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3t^2 & 2t \\ 0 & 6t & 2 \end{vmatrix} = (6t^2 - 12t^2)\,\hat{i}$

$\therefore \kappa(t) = \dfrac{|6t^2 - 12t^4|}{\sqrt{(3t^2)^2 + (2t)^2}} = \boxed{\dfrac{6t^2}{\sqrt{9t^4 + 4t^2}}}$ $\neq$ ~~$6t^2$~~

Free Response. Show your work!

3. (10 points) Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

if it exists or, if it does not exist, explain why.

Let $y = mx$, $f(x,y) = \dfrac{xy}{x^2+y^2}$. Then

$f(x,mx) = \dfrac{mx^2}{x^2+m^2x^2} = \dfrac{m}{1+m^2}$. Hence f has constant, distinct values along lines of different slope, so e.g. $\lim_{x\to 0} f(x,x) = \frac{1}{2}$ while $\lim_{x\to 0} f(x,0) = 0$. Hence, the limit DNE

4. (10 points) Find $f_{xx}(2,-1)$ and $f_{xy}(2,-1)$, if

$$f(x,y) = \frac{y}{2x+3y}.$$

$$\boxed{f_x(x,y) = -\frac{2y}{(2x+3y)^2}}$$

$$f_{xx}(x,y) = \frac{+2\cdot 2\cdot 2y}{(2x+3y)^3} \qquad f_{xy}(x,y) = \frac{-2}{(x+3y)^2} + \frac{2(2y)\cdot 3}{(x+3y)^3}$$

$$\cancel{f_{xx}(2,-1) = -2\cdot 3\cdot(-1)}$$

$$\boxed{f_{xx}(x,y) = \frac{8y}{(2x+3y)^3}} \qquad \boxed{f_{xy}(x,y) = \frac{-2}{(2x+3y)^2} + \frac{12y}{(2x+3y)^3}}$$

$$f_{xx}(2,-1) = \frac{-8}{1} = \boxed{-8}$$

$$f_{xy}(2,-1) = -\frac{2}{1^2} + \frac{-12}{1^3} = \boxed{-14}$$

5. (10 points) Find the linearization of $f(x, y) = 12\arctan(xy)$ at $(1, 1)$. Exact answer is expected.

$$f_x(x,y) = \frac{12y}{1+x^2y^2} \qquad f_y = \frac{12x}{1+x^2y^2}$$

$$f_x(1,1) = \frac{12}{13} \qquad f_y(1,1) = \frac{12}{13}$$

$$L(x,y) = 12\arctan(1) + \frac{12}{13}(x-1) + \frac{12}{13}(y-1)$$

$$= \cancel{12}^{3}\cdot\frac{\pi}{\cancel{4}} + \frac{12}{13}(x-1) + \frac{12}{13}(y-1)$$

6. (10 points) Let

$$w = xy + yz + zx, \quad x = r\cos\theta, \quad y = r\sin\theta, \quad z = r\theta.$$

Use the chain rule to evaluate $\partial w/\partial r$ and $\partial w/\partial \theta$ when $r = 2$ and $\theta = \pi/2$.

$$\frac{\partial w}{\partial x} = y+z \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial z}{\partial r} = \theta$$

$$\frac{\partial w}{\partial y} = x+z \qquad \frac{\partial y}{\partial \theta} = r\cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial z}{\partial \theta} = r$$

$$\frac{\partial w}{\partial z} = x+y$$

For $r=2$, $\theta = \pi/2$, $\boxed{x = 0,\ y = 2,\ z = \pi}$ $\boxed{\frac{\partial x}{\partial \theta} = -2,\ \frac{\partial y}{\partial \theta} = 0,\ \frac{\partial x}{\partial r} = 0,\ \frac{\partial y}{\partial r} = 1}$ $\frac{\partial z}{\partial r} = \frac{\pi}{2}$, $\frac{\partial z}{\partial \theta} = 2$

$$\frac{\partial w}{\partial \theta} = (y+z)\frac{\partial x}{\partial \theta} + (x+z)\frac{\partial y}{\partial \theta} + (x+y)\frac{\partial z}{\partial \theta}$$

$$= (2+\pi)(-2) + (\pi)\cdot 0 + (2)\cdot 2$$

$$= \boxed{-2\pi}$$

$$\frac{\partial w}{\partial r} = (y+z)\frac{\partial x}{\partial r} + (x+z)\frac{\partial y}{\partial r} + (x+y)\frac{\partial z}{\partial r}$$

$$= (2+\pi)\cdot 0 + \pi\cdot 1 + 2\cdot\frac{\pi}{2} = \boxed{2\pi}$$

Free Response. Show your work!

7. (10 points) Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$, if $e^z = xyz$.

$$e^z \frac{\partial z}{\partial x} = yz + xy\frac{\partial z}{\partial x} \Rightarrow (e^z - xy)\frac{\partial z}{\partial x} = yz$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}}$$

$$e^z \frac{\partial z}{\partial y} = xz + xy\frac{\partial z}{\partial y}$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}}$$

8. (10 points) Find an equation for the tangent plane to the surface $xy^2z^3 = 8$ at $(2, 2, 1)$. Write the equation in the form $x + by + cz = d$.

$$y^2z^3 + xy^2 \cdot 3z^2 \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-y^2z^3}{3xy^2z^2} = \frac{-z}{3x}$$

$$2xyz^3 + xy^2 \cdot 3z^2\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-2xyz^3}{3xy^2z^2} = -\frac{2z}{3y}$$

$$\therefore f_x(2,2) = -\frac{1}{3\cdot 2} \qquad f_y(2,2) = \frac{-2}{6} = -\frac{1}{3}$$

$$= -\frac{1}{6}$$

$$\therefore z = 1 + -\frac{1}{6}(x-2) + -\frac{1}{3}(y-2)$$

$$\therefore z = 1 + \frac{1}{3} + \frac{2}{3} - \frac{1}{6}x - \frac{1}{3}y$$

$$z = 2 - \frac{1}{6}x - \frac{1}{3}y$$

$$\therefore \frac{1}{6}x + \frac{1}{3}y + z = 2$$

$$\therefore \boxed{x + 2y + 6z = 12}$$

9. (10 points) Find the critical points of $f(x, y) = x^3 - 3x + y^4 - 2y^2$ and classify each of them as local maximum, local minimum, or saddle point.

$f_x(x,y) = 3x^2 - 3 = 3(x^2-1) \Rightarrow x = \pm 1$

$f_y(x,y) = 4y^3 - 4y = 4y(y^2-1) \Rightarrow y = \pm 1, 0$

$f_{xx} = 6x \quad f_{xy} = 0 \quad f_{yy} = 12y^2 - 4$

$\text{Hess}(f) = \begin{pmatrix} 6x & 0 \\ 0 & 12y^2-4 \end{pmatrix} \qquad D = 6x(12y^2-4)$

c.p.	D	f_{xx}	Type
(1,−1)	48	6	min
(1,0)	−24		saddle
(1,1)	48	6	min
(−1,−1)	−48		saddle
(−1,0)	24	−6	max
(−1,1)	−48		saddle

Free Response. Show your work!

10. (10 points) Find the absolute maximum and absolute minimum values of $f(x,y) = y^2 - x^2$ on the unit disk $D = \{(x,y) \mid x^2 + y^2 \leq 1\}$.

$f_x = -2x \qquad f_{xx} = -2 \qquad f_{xy} = 0$

$f_y = 2y \qquad f_{yy} = 2$

At $(0,0)$, the only critical pt, $D = -4$ so saddle pt.

On the bdry, parameterize

$x = \cos t$

$y = \sin t$

$\varphi(t) = f(x(t), y(t)) = \sin^2 t - \cos^2 t \qquad 0 \leq t \leq 2\pi$

$\varphi'(t) = 2\sin t \cos t + 2\cos t \sin t$

$= 4 \sin t \cos t$

$\varphi'(t) = 0$ at $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

t	$\cos t$	$\sin t$	$\sin^2 t - \cos^2 t$
0	1	0	-1
$\pi/2$	0	1	1
π	-1	0	-1
$3\pi/2$	0	-1	1
2π	1	0	-1

Absolute max: $+1$

Absolute min: -1