Math 575 - Principles of Analysis

Midterm Exam I

Name: KEY

1. (a) (5 points) Say what it means for a real number c to be the least upper bound of a subset A of $\mathbb{R}$.

   Solution: c is the least upper bound of A if:
   (2 points) c is an upper bound of A and
   (3 points) $c \leq c'$ for any upper bound c' of A.

   (b) (20 points) Suppose that A and B are subsets of $\mathbb{R}$ and that A and B are bounded above. Define

   $$A + B = \{x + y : x \in A, y \in B\}.$$

   Prove that

   $$\sup(A + B) = \sup(A) + \sup(B)$$

   Solution: (5 points) For every $a \in A$ and $b \in B$, we have $a \leq \sup(A)$ and $b \leq \sup(B)$. (5 points) Hence

   $$a + b \leq \sup(A) + \sup(B)$$

   which shows that the number $c = \sup(A) + \sup(B)$ is an upper bound for $A + B$.

   (10 points) Proof 1: Given any $\varepsilon > 0$ there are numbers $a \in A$ and $b \in B$ so that $a > \sup(A) - \varepsilon$ and $b > \sup(B) - \varepsilon$, so that $a + b \geq c - 2\varepsilon$. This shows that $c - \varepsilon$ is not an upper bound for any $\varepsilon > 0$, so c is the least upper bound.

   Proof 2 (courtesy of Ethan Reed): Suppose that c is an upper bound for $A + B$. If $a \in A$ and $b \in B$ then $a + b \leq c$ or $a \leq c - b$ for all $a \in A$. It follows that $\sup(A) \leq c - b$ so that $\sup(A) + b \leq c$ for all $b \in B$. Hence, $\sup(A) + \sup(B) \leq c$, proving that $\sup(A) + \sup(B)$ is the least upper bound.

2. (a) (5 points) Say what it means for a sequence $\{a_n\}$ of complex numbers to be Cauchy.

   Solution: (5 points) A sequence $\{a_n\}$ is Cauchy if, given any $\varepsilon > 0$, there is an $n \in \mathbb{N}$ so that $|a_n + m - a_n| < \varepsilon$ whenever $n \geq N$ and $m \in \mathbb{N}$.

   (b) (20 points) Suppose that $\{a_n\}$ is a complex sequence, that $\{b_n\}$ is a nondecreasing sequence of positive numbers which converge to a limit, and that

$$|a_{n+1} - a_n| \leq b_{n+1} - b_n.$$

   Show that $\{a_n\}$ is a Cauchy sequence.

   Solution: (10 points) Observe that

$$\begin{aligned}
|a_{n+m} - a_n| &\leq \sum_{j=1}^{m} |a_{n+j} - a_{n+j-1}| \\
&\leq \sum_{j=1}^{m} (b_{n+j} - b_{n+j-1}) \\
&= b_{n+m} - b_n \\
&= |b_{n+m} - b_n|
\end{aligned}$$

   where the last step follows because $\{b_n\}$ is nondecreasing.
   (10 points) Since $\{b_n\}$ is Cauchy, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ so that $|b_{n+m} - b_n| < \varepsilon$ for all $n \geq N$ and $m \in \mathbb{N}$. It then follows that $|a_{n+m} - a_n| < \varepsilon$ for all such m, n, which proves that $\{a_n\}$ is Cauchy.

3. (25 points) Consider the series $\sum_{n=1}^{\infty} a_n$ and suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1.$$

   Prove directly (i.e., without appealing to the ratio test) that $\sum_{n=1}^{\infty} a_n$ is convergent. You may assume that the comparison test holds.

Solution: (5 points) Let $r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ and choose s with $r < s < 1$.

(5 points) By the definition of limit, there is a positive integer N so that $\left|\frac{a_{n+1}}{a_n}\right| < s$ for all $n \geq N$.

(5 points) By iteration we conclude that $|a_{n+m}| \leq s^m |a_n|$ for all $m \in \mathbb{N}$.

(10 points) We can then conclude from the comparison test that the series $\sum_{m=1}^{\infty} |a_{n+m}|$ converges since the geometric series $\sum_{m=1}^{\infty} s^n$ converges.

4. (a) (5 points) Define $\limsup_{n\to\infty} x_n$ for a bounded sequence of real numbers $\{x_n\}$.

   Solution: (5 points) Let $a_m = \sup\{x_n : n \geq m\}$. Then

$$\limsup_{n\to\infty} x_n = \lim_{m\to\infty} a_m$$

   where the latter limit exists because $\{a_m\}$ is nondecreasing.

   (b) (15 points) Prove that if $\{x_n\}$ and $\{y_n\}$ are bounded sequences, then

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$

   Solution: (5 points) Let $A = \limsup_{n\to\infty} x_n$ and $B = \limsup_{n\to\infty} y_n$. Let

$$a_m = \sup\{x_n : n \geq m\},$$
$$b_m = \sup\{y_n : n \geq m\}.$$

   Then $a_m \to A$ and $b_m \to B$ as $m \to \infty$.

   (5 points) Given any $\varepsilon > 0$, we can find an N_1 so that $a_m \leq A + \varepsilon$ for all $m \geq N_1$, and an N_2 so that $b_m \leq B + \varepsilon$ for all $m \geq N_2$. Choosing $N = \max(N_1, N_2)$ we have that $a_m + b_m \leq A + B + 2\varepsilon$ for all $m \geq N$.

   (5 points) From problem 1 (!), we have

$$\sup\{x_n + y_n : n \geq m\} \leq a_m + b_m \leq A + B + 2\varepsilon$$

from which it follows that

$$\limsup_{n\to\infty}(x_n + y_n) \leq A + B + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the desired inequality.

(c) (5 points) Give an example of sequences $\{x_n\}$ and $\{y_n\}$ where strict inequality holds.

Solution: (5 points) There are lots of possible examples. Here's one that was a common choice. Let

$$x_n = (-1)^n, \quad y_n = (-1)^{n+1}.$$

Then $x_n + y_n = 0$ for every n but $\limsup x_n = \limsup y_n = 1$, so that the sum is 2. The underlying idea is that there should be some kind of cancellation in the sum $x_n + y_n$.