PRINCIPLES OF ANALYSIS - LECTURE NOTES

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2. Real and Complex Numbers

If A is a subset of $\mathbb{R}$, we say that A is *bounded above* if there is a real number b so that $a \leqslant b$ for all $a \in A$. Similarly, A is *bounded below* if there is a real number c so that $c \leqslant a$ for all $a \in A$.

An upper bound b is called a *least upper bound* for A if $b \leqslant b'$ for every upper bound b' of b. A lower bound is called a *greatest lower bound* for A and $c' \leqslant c$ or every lower bound c' of A.

The following property of $\mathbb{R}$ will play a fundamental role in analysis.

O6 If A is a nonempty subset of $\mathbb{R}$ that is bounded above, there is a least upper bound for A.

In fact, the least upper bound and greatest lower bound are unique (why?). The least upper bound of a set A is denoted $\sup A$, and the greatest lower bound of A is denoted by $\inf A$.

Complex numbers are numbers of the form $x+iy$ where x and y are real numbers and i is an algebraic object with the property that $i^2 = -1$. Complex numbers $z = x + iy$ are in one-to-one correspondence with points $(x, y) \in \mathbb{R}^2$. We denote $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. If $z = x + iy$, the complex number $\overline{z}$ is $x - iy$, and the real number $|z| = \sqrt{x^2 + y^2}$ is called the *modulus* of z.

There is also an analogue of 'polar coordinates' in $\mathbb{C}$. For $z \neq 0$, write

☞This decomposition was discussed in class during group work

$$z = |z| \cdot \frac{z}{|z|}.$$

The scalar $|z|$ is the distance of z from 0, while $z/|z|$ is a complex number of unit modulus which specifies the 'direction' of z. We may write

$$z = rw$$

where $r = |z|$ is a positive real number, and $w = z/|z|$ is a complex number of modulus 1.

The following inequalities will play an important role in the study of limits of sequences.

$$\begin{aligned} |\operatorname{Re} z| &\leqslant |z| \\ |\operatorname{Im} z| &\leqslant |z| \\ |z| &\leqslant |\operatorname{Re} z| + |\operatorname{Im} z| \\ |z + w| &\leqslant |z| + |w| \end{aligned} \tag{2.1}$$

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For legal purposes, be it duly noted that these notes are a gloss on the text *Introduction to Analysis* by Richard Beals. Therefore no originality is claimed for the presentation! Worse still, some of the proofs are lifted directly out of the text with trivial modifications.

These notes are for student use only in MA 575 Fall 2018 at the University of Kentucky.

3. Real and Complex Sequences

A *sequence* from $\mathbb{R}$ or $\mathbb{C}$ is a list of numbers $\{z_n\}$ in one-to-one correspondence with the positive integers. The following definition is fundamental.

Definition 3.1. The complex sequence $\{a_n\}$ *converges* if there is a complex number a with the property that, for each $\varepsilon > 0$, there is an integer N so that $n \geqslant N$ implies that $|a_n - a| < \varepsilon$. The number a is called the *limit* of the sequence $\{a_n\}$, and we write

$$a = \lim_{n \to \infty} a_n.$$

Proposition 3.2. *A complex sequence $a_n + ib_n$ has limit $a + ib$ if and only if the real sequences $\{a_n\}$ and $\{b_n\}$ have limits a and b, respectively.*

☞ This proof was worked out in class during group work

Proof. We'll use the inequalities (2.1).

(⇒) Suppose that $a_n + ib_n \to a + ib$. Then for every $\varepsilon > 0$, there is an N so that

$$|(a_n + ib_n) - (a + ib)| < \varepsilon.$$

Fro the first and second inequalities of (2.1), we conclude that

$$|a_n - a| < \varepsilon, \quad |b_n - b| < \varepsilon.$$

Hence $a_n \to a$ and $b_n \to b$.

(⇐): Suppose that $a_n \to a$ and $b_n \to b$. Then, for every $\varepsilon > 0$, there is an N so that for all $n \geqslant N$,

$$|a_n - a| < \varepsilon/2, \quad |b_n - b| < \varepsilon/2$$

By the third inequality of (2.1),

$$|(a_n + ib_n) - (a + ib)| = |(a_n - a) + i(b_n - b)| \leqslant |a_n - a| + |b_n - b| < \varepsilon$$

so that $a_n + ib_n \to a + ib$. □

3.1. Monotone Sequences.

A real sequence $\{a_n\}$ is called *nondecreasing* if

$$a_1 \leqslant a_2 \leqslant \ldots \leqslant a_{n-1} \leqslant a_n$$

for every n. Such a sequence is called *nonincreasing* if

$$a_1 \geqslant a_2 \geqslant \ldots \geqslant a_{n-1} \geqslant a_n$$

for every n.

A sequence which is either nondecreasing or nonincreasing is called *monotone*. A nondecreasing subsequence is bounded below, but not necessarily bounded above. A nonincreasing subsequence is bounded above, but not necessarily bounded below.

The least upper bound property implies the following fundamental result about convergence.

Theorem 3.3. *Any bounded monotone sequence of reals is convergent.*

Proof. We'll give the proof for a monotone nonincreasing sequence (just for a change of pace from the text). Denote by $\{a_n\}$ the sequence and let $a = \inf(\{a_n\})$. We claim that $a_n \to a$. Let $\varepsilon > 0$ be given. Since $a + \varepsilon$ is not a lower bound for $\{a_n\}$, there is at least one a_N with $a_N < a + \varepsilon$. Since $\{a_n\}$ is nonincreasing, $a \leqslant a_n < a + \varepsilon$ for all $n \geqslant N$. Hence $a_n \to a$. □

3.2. **Lim sup and lim inf.** Now suppose that $\{x_n\}$ is a bounded sequence of real numbers. We define two new sequences associated to $\{x_n\}$ by

$$a_m = \inf\{x_m, x_{m+1}, x_{m+2}, \ldots\}$$
$$b_m = \sup\{x_m, x_{m+1}, x_{m+2}, \ldots\}$$

The sequence $\{a_m\}$ is monotone nondecreasing (deleting elements from the set can only make its lower bound larger) while $\{b_m\}$ is monotone nonincreasing (deleting elements from the set can only make its upper bound smaller). This means that the intervals $I_m = [a_m, b_m]$ are nested so that in particular, the sequences $\{a_m\}$ and $\{b_m\}$ are bounded as well as monotone. Hence, they both converge. We *define*

$$\liminf x_n = \lim_{n\to\infty} a_n$$
$$\limsup x_n = \lim_{n\to\infty} b_n$$

Clearly $\liminf x_n \leqslant \limsup x_n$.

Theorem 3.4. $\liminf x_n = \limsup x_n$ *if and only if* x_n *converges.*

To prove the theorem we'll given an alternative characterization of the numbers $\liminf x_n$ and $\limsup x_n$ that lends itself nicely to the proof.

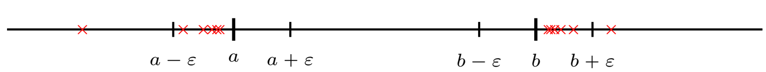
Proposition 3.5. *Suppose that* $\{x_n\}$ *is a bounded sequence of real numbers.*

(a) $\liminf x_n = a$ *if and only if*
  (i) *For every* $\varepsilon > 0$*, there are at most a finite number of* x_n *with* $x_n \leqslant a - \varepsilon$
  (ii) *For every* $\varepsilon > 0$*, there are infinitely many* x_n *with* $x_n < a + \varepsilon$*.*
(b) $\limsup x_n = b$ *if and only if*
  (i) *For every* $\varepsilon > 0$*, there are at most a finite number of* x_n *with* $x_n \geqslant b + \varepsilon$
  (ii) *For every* $\varepsilon > 0$*, there are infinitely many* x_n *with* $x_n > b - \varepsilon$

Remark 3.6. A good way to think about the characterization of lim inf and lim sup above is to think of a specific sequence and its graph on the real line. The sequence

$$x_n = (-1)^n + \frac{(-1)^n}{n}$$

has two cluster points, one at $a = -1$ (the lim inf) and the other at $b = +1$ (the lim sup). Its graph looks like



Notice that there is only one 'outlier' to the left of $a - \varepsilon$, and similarly only one 'outlier' to the right of $b + \varepsilon$. On the other hand, there are infinitely many elements of the sequence to the left of $a + \varepsilon$, and infinitly many to the right of $b - \varepsilon$.

Proof. We'll prove both directions for (a).

($\Rightarrow$) If $\liminf x_n = a$, there is an N so that $a - \varepsilon < a_N < a$. Since $x_n \geqslant a_N$ for all $n \geqslant N$, this means that at most $x_1, x_2, \ldots x_{N-1}$ lie to the left of $a - \varepsilon$, proving (i). On the other hand, if there were only finitely many n (say, up to x_M, with $x_n < a + \varepsilon$, we would then have $a_{M+1} \geqslant a + \varepsilon$, which can't occur. Hence there are infinitely many such x_n, proving (ii).

($\Leftarrow$) Suppose that a is a real number satisfyng (i) and (ii). Statement (i) implies that, given any $\varepsilon > 0$, there is an N so that $x_n \geqslant a - \varepsilon$ for all $n \geqslant N$, and hence

$a_n \geqslant a - \varepsilon$ for all such n. This shows that $\liminf x_n \geqslant a - \varepsilon$ for any $\varepsilon > 0$. On the other hand, statement (ii) implies that $a_N < a + \varepsilon$ for all sufficiently large n. Since $a - \varepsilon \leqslant a_n < a + \varepsilon$ for all sufficiently large n, it follows that $a - \varepsilon \leqslant \lim a_n < a + \varepsilon$. Since $\limsup x_n = \lim a_n$ and $\varepsilon > 0$ is arbitrary, $\limsup a_n = a$. □

Now we can prove the main theorem.

Proof of Theorem 3.4. ($\Leftarrow$) Suppose that $\lim x_n = x$. For any $\varepsilon > 0$ there is a positive integer N so that $x - \varepsilon \leqslant x_n \leqslant x + \varepsilon$ for all $n \geqslant N$. Thus $x_n \leqslant x - \varepsilon$ for at most finitely many n, while $x_n < x + \varepsilon$ for infinitely many n. Hence $x = \liminf x_n$. On the other hand, given $\varepsilon > 0$, we can use the same fact to conclude that $x_n > x + \varepsilon$ for at most finitely many n, while $x_n > x - \varepsilon$ for infinitely many n. Hence $x = \limsup x_n$.

☞This paragraph clarifies an obscure point from the in-class presentation

($\Rightarrow$) Suppose that $\liminf x_n = \limsup x_n = x$. For any given $\varepsilon > 0$, there are at most finitely many n with $x_n > x + \varepsilon/2$ or $x_n < x - \varepsilon/2$ by properties a(i) and b(i). Hence, for some N, $x - \varepsilon < x_n < x + \varepsilon$ for all $n \geqslant N$.

□