Math 575
Fall 2018
Solutions to Problem Set # 10

(1) (p. 91, 1) Suppose that f obeys

$$f(x+y) = f(x) + f(y)$$

Clearly $f(2x) = 2f(x)$ (by taking $x = y$). We claim that, also $f(2^n x) = 2^n f(x)$. Given that this identity holds for all $n \leq N$. we compute

$$\begin{aligned} f(2^{N+1}x) &= f(2^N x) + f(2^N x) \\ &= 2f(2^N x) \\ &= 2^{N+1} f(x). \end{aligned}$$

We seek to show that $f(x) = ax$ where $a = f(1)$. The basic identity extends to show that

$$f\left(\sum_{i=1}^{N} x_i\right) = \sum_{i=1}^{N} f(x_i).$$

Moreover, since $f(2^N x) = 2^n f(x)$, it follows that

$$f(2^{-j}) = 2^{-j} f(1)$$

by taking $N = j$ and $x = 2^{-j}$. Hence, for any *dyadic rational* in $[0,1]$, i.e., any number x that can be written as

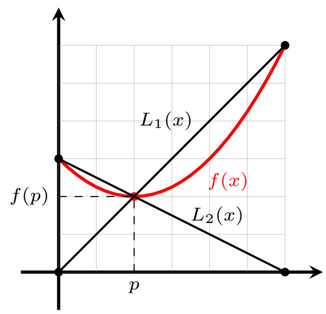
$$x = \sum_{i=1}^{n} \frac{a_i}{2^i}$$

where $a_i = 0$ or 1,

$$f(x) = xf(1).$$

A similar argument shows that $(x) = xf(1)$ for $x \in [-1,0]$. Next, any dyadic rational x in $\mathbb{R}$ can be written as $2^m y$ for some $y \in [-1,1]$, so we can conclude that $f(x) = xf(1)$ for all dyadic rationals. Since the dyadic rationals are dense in $\mathbb{R}$, it then follows by continuity that $f(x) = xf(1)$ for all real x.

(2) (p. 91, 3) The following very nice proof is due to a former student, Kristina Pepe.



We'll show that f is continuous at p using the "squeeze theorem." We'll use the convexity of f to show that the inequalities

$$\begin{aligned} L_1(x) \le f(x) \le L_2(x), \quad x \in (p-r, p) \\ L_2(x) \le f(x) \le L_1(x), \quad x \in (p, p+r) \end{aligned} \tag{1}$$

hold for some fixed $r > 0$, where

$$L_1(x) = \frac{f(p+r) - f(p)}{r}(x-p) + f(p),$$
$$L_2(x) = \frac{f(p) - f(p-r)}{r}(x-p) + f(p).$$

If the inequalities (1) hold, then

$$\lim_{x \to p^-} f(x) = \lim_{x \to p^+} f(x) = f(p)$$

which shows that f is continuous at p.

Let's prove the inequalities (1). For $x \in (p, p+r)$ we write

$$x = (1-t)p + t(p+r)$$

for some $t \in (0,1)$. In fact, $t = (x-p)/r$. By convexity

$$\begin{aligned} f(x) &\le \left(1 - \frac{x-p}{r}\right) f(p) + \frac{x-p}{r} f(p+r) \\ &= L_1(p) \end{aligned}$$

A similar proof shows that $L_2(x) \le f(x)$ for $x \in (p-r, p)$.

To show the remaining inequalities we use, believe it or not, proof by contradiction. To show that $L_2(x) < f(x)$ for $x \in (p, p+r)$, suppose to the contrary that $L_2(x_0) \ge f(x_0)$ for some $x_0 \in (p, p+r)$. Since $p-r < p < x_0$, we may write

$$p = (1-t)(p-r) + tx_0, \quad t = \frac{r}{x_0 - p + r}$$

and use convexity of f to conclude that

$$f(p) \le \frac{x_0 - p}{x_0 - p + r} f(p-r) + \frac{r}{x_0 - p + r} f(x_0). \tag{2}$$

On the other hand, if $f(x_0) < L_2(x_0)$, it follows from the definition of $L_2(x)$ that

$$f(x_0) < \frac{f(p) - f(p-r)}{r}(x_0 - p) + f(p)$$

or

$$f(x_0) + \frac{x_0 - p}{r} f(p-r) < \frac{x_0 - p + r}{r} f(p)$$

so that, dividing both sides by the positive number $(x_0 - p + r)/r$, we obtain

$$\frac{r}{x_0 - p + r} f(x_0) + \frac{x_0 - p}{x_0 - p + r} f(p-r) < f(p)$$

which contradicts (2). Hence, $L_2(x) < f(x)$ for $x \in (p, p+r)$.

A similar proof shows that $L_1(x) < f(x)$ for $x \in (p-r, p)$.

(3) (p. 94, 1) This problem deserves its own two pages!

In what follows we will use the fact that if $\{a_n\}$ is a sequence of positive real numbers and $\log a_n \to -\infty$, then $a_n \to 0$. We also make use of the "compound interest formula"

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{nb} = e^{ab}.$$

(a) Let $f_n(x) = nx^2(1-x)^n$. Note that $f_n(0) = f_n(1) = 0$. For $0 < x < 1$, we compute

$$\log f_n(x) = n\left(\frac{\log n}{n} + \log(1-x)\right) \to -\infty$$

so $f_n(x) \to 0$ as $n \to \infty$ for all $x \in [0,1]$. Thus $f_n(x) \to 0$ uniformly if and only if $M_n = \sup_{x\in[0,1]} |f_n(x)|$ converges to zero. Since

$$\frac{f_n'(x)}{f(x)} = \frac{2}{x} + \frac{n}{1-x}$$

f has a unique critical point at $x = 2/(n+2)$. Thus

$$M_n = \frac{4n}{(n+2)^2}\left(1+\frac{2}{n}\right)^n$$

which tends to zero since the first right-hand factor tends to zero and the second right-hand factor converges to e^{-2}. Hence $f_n \to 0$ uniformly.

(b) Let $f_n(x) = n^2x(1-x^2)^n$. Again $f_n(0) = f_n(1) = 0$ while for $0 < x < 1$,

$$\log f_n(x) = n\left(\frac{2\log n}{n} + \frac{\log x}{n} + \log(1-x^2)\right) \to -\infty$$

so $f_n(x) \to 0$ as $n \to \infty$ for all x. For this function

$$\frac{f_n'(x)}{f(x)} = \frac{1}{x} - \frac{2nx}{1-x^2}$$

so the unique critical point in $(0,1)$ occurs at $x_n = \sqrt{2n+1}$. A computation shows that

$$M_n - \frac{n^2}{2n+1}\left(1+\frac{1}{2n}\right)^{-n}.$$

The first factor diverges while the second factor converges to $e^{-1/2}$. Hence f_n does not converge uniformly to 0.

(c) Let $f_n(x) = n^2x^3e^{-nx^2}$. Then $f_n(0) = 0$ and $f_n(1) = n^2e^{-n} \to 0$ as $n \to \infty$. For $0 < x < 1$ we have

$$\log f_n(x) = n\left(\frac{2\log n}{n} + \frac{3\log x}{n} - x^2\right) \to -\infty$$

so $f_n(x) \to 0$ as $n \to \infty$ for all $x \in [0,1]$. Since

$$\frac{f_n'(x)}{f_n(x)} = \frac{3}{x} - 2nx$$

f has a unique critical point at $x_n = (3/(2n))^{1/2}$. Thus

$$f_n(x_n) = n^2 \left(\sqrt{\frac{3}{2n}} \right)^3 e^{-3/2}$$

which diverges as $n \to \infty$. Hence f_n does not converge to zero uniformly.

(d) Let $f_n(x) = \dfrac{x^2}{x^2 + (1 - nx)^2}$. Then $f_n(0) = 0$ and $f_n(1) = \frac{1}{1+(n-1)^2} \to 0$ as $n \to \infty$. For $0 < x < 1$ we have

$$f_n(x) \leq \frac{x^2}{(1 + n^2)\left(x^2 - \frac{2nx}{1+n^2} \right)} \to 0 \text{ as } n \to \infty$$

so that $f_n(x) \to 0$ for all $x \in [0, 1]$. To find the interior critical point we use the quotient rule to compute

$$f_n'(x) = (-2) \frac{(nx - 1)x}{(x^2 - 2nx + n^2x^2 + 1}$$

so that $x_n = 1/n$. We compute $f_n(x_n) = 1$ which shows that $f_n(x)$ does not converge to 0 uniformly.

(4) (p. 94, 2)

(a) Suppose that $\{f_n\}$ is a sequence of functions from A which converges in $C(I)$ to a limit function f. Since $|f_n(x)| \leq 1$ and $f_n(x) \to f(x)$ as $n \to \infty$ for each x, it follows that $|f(x)| \leq 1$ for all $x \in I$, so that $f \in A$. Hence A is closed.
Second, for any $f \in A$. $|f(x)| \leq 1$ for all $x \in I$, so that $|f| \leq 1$. Hence, A is a bounded set.

(b) Let $f_n(x) = x^n$ and consider the function

$$g_{n,m}(x) = |f_{n+m}(x) - f_n(x)| = x^n\left(1 - x^m\right).$$

Note that $g_{n,m}(x) \geq 0$ and $g_{n,m}(0) = g_{n,m}(1) = 0$, so the maximum occurs at an interior critical point. To find it we compute

$$\frac{g'_{n,m}(x)}{g_{n,m}(x)} = \frac{n}{x} - \frac{mx^{m-1}}{1-x^m}$$

so that the critical point is $x_{n,m} = (n/(n+m))^{1/m}$. Thus

$$g_{n,m}(x_{n,m}) = \left(\frac{n}{n+m}\right)^{n/m}\left(\frac{m}{n+m}\right).$$

As $m \to \infty$, the second factor goes to 1 while the first factor also goes to 1 as $m \to \infty$ for fixed n since

$$\left(\frac{n}{n+m}\right)^{n/m} = e^{(n/m)\log(1-m/(n+m))}.$$

This shows that $\|f_{n+m} - f_n\| \to 1$ as $m \to \infty$.

(5) (p. 94, 6)

(a) The set A of complex numbers of the form $p+iq$ with $p, q \in \mathbb{Q}$ is in one-to-one correspondence with $\mathbb{Q} \times \mathbb{Q}$ which, as a finite cartesian product of countable sets, is countable. Given $z = x + iy$ there are sequences $\{p_n\}$ and $\{q_n\}$ from $\mathbb{Q}$ so that $p_n \to x$, $q_n \to y$ as $n \to \infty$. Thus given any z and $\varepsilon > 0$, we can find $p + iq \in A$ with $|z - (p + iq)| < \varepsilon$. This shows that A is dense in $\mathbb{C}$.

(b) Fix an interval $I = [a, b]$. We will show that polynomials with rational coefficients form a countable dense subset of $C(I)$.
First, observe that he set $\mathcal{Q}_n$ of polynomials of degree n with rational coefficients is countable because it lies in one-to-one correspondence with $\mathbb{Q}^{n+1}$. Since the set of rational polynomials $\mathcal{Q}$ is given by $\mathcal{Q} = \cup_{n=0}^{\infty}\mathcal{Q}_n$, it follows that $\mathcal{Q}$ is countable.

Next, note that, given $\varepsilon > 0$ and a polynomial P_n is a polynomial with real coefficients, there is a $Q_n \in \mathcal{Q}_n$ with $\|P_n - Q_n\| < \varepsilon/2$. This follows from the fact that, for any polynomials $P = \sum_{j=0}^{n} \alpha_j x^j$ and $Q = \sum_{j=0}^{n} \beta_j x^j$,

$$\|P_n - Q_n\| \leq \sum_{j=1}^{n} |\beta_j - \alpha_j| \max(|a|, |b|)^j.$$

Finally, given any $\varepsilon > 0$ and any $f \in C(I)$, we can find a polynomial P_n with $\|f - P\| < \varepsilon/2$. by the Weierstrass polynomial approximation theorem.

Combining these observations, we see that, given any $\varepsilon > 0$ and an $f \in C(I)$, there is a polynomial Q_n with rational coefficients such that $\|f - Q_n\| < \varepsilon$.