Math 575
Problem Set #11
Solutions

1. (p. 104, 1b) To find

$$\lim_{x \downarrow 0} \frac{\log(1+x) - x}{\sin(x^2)},$$

let $f(x) = \log(1+x) - x$, and let $g(x) = \sin(x^2)$. By continuity, $\lim_{x\downarrow 0} f(x) = \lim_{x\downarrow 0} g(x) = 0$. Note that g is nonvanishing in $(0,1)$. Since

$$f'(x) = \frac{1}{1+x} - 1$$
$$g'(x) = 2x\cos\left(x^2\right)$$

we see that $g'(x)$ is nonvanishing in $(0,1)$ and $\lim_{x\downarrow 0} f'(x) = \lim_{x\downarrow 0} g'(x) = 0$. Consider

$$f''(x) = -\frac{1}{(1+x)^2},$$
$$g''(x) = 2\cos\left(x^2\right) - 4x^2\sin(x^2).$$

Note that $g''(x) \neq 0$ for $x > 0$ sufficiently small. By continuity and the algebra of limits,

$$\lim_{x\downarrow 0} \frac{f''(x)}{g''(x)} = \frac{1}{2}.$$

It follows from L'Hospital's rule that, also

$$\lim_{x\downarrow 0} \frac{f'(x)}{g'(x)} = \frac{1}{2},$$

and with one more application we get

$$\lim_{x\downarrow 0} \frac{f(x)}{g(x)} = \frac{1}{2}.$$

2. (p. 104, 2) We apply the Mean Value Theorem. For any $x, y \in \mathbb{R}$ there is a point $c \in (x,y)$ so that

$$\frac{f(x) - f(y)}{x - y} = f'(c)$$

so

$$|f(x) - f(y)| \leq |f'(c)|\,|x - y|.$$

Since f' is bounded in $\mathbb{R}$, there is a positive number M so that $|f'(c)| \leq M$ for all $c \in \mathbb{R}$. Hence

$$|f(x) - f(y)| \leq M\,|x - y|.$$

Given any $\varepsilon > 0$, choosing $\delta < \varepsilon/M$ gives $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. This shows that f is uniformly continuous.

3. (p. 104, 3) (a) Let $\varepsilon > 0$ be given, and choose $M \geq 1$ so that $|f(x)| < \varepsilon/2$ for $|x| \geq M$. Since f is continuous on the compact interval $[-2M, 2M]$, it is uniformly continuous on $[-2M, 2M]$. There is a $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon/2$ whenever $|x - y| < \delta$ and $x, y \in [-M, M]$. We claim that, for *any* x, y with $|x - y| < \delta$, the same holds true. Either both of x and y lie in $[-M, M]$, at least one of x, y lie in $[-M, M]$, or neither of x, y lie in $[-M, M]$. In the first case, the result is already proved, and in the third case $|f(x) - f(y)| \leq |f(x)| + |f(y)| < \varepsilon$. In the second case, if say $x \in [-M, M]$ but $y \notin [-M, M]$, we still have $y \in [-2M, 2M]$ so $|f(x) - f(y)| < \varepsilon$.
(b) Consider the function

$$f(x) = \frac{\cos\left(x^4\right)}{1 + x^2}.$$

Since $\left|\cos\left(x^4\right)\right| \leq 1$, it is easy to see that $\lim_{|x| \to \infty} f(x) = 0$, so by part (a) f is uniformly continuous on the real line. On the other hand,

$$f'(x) = \frac{4x^3 \sin(x^4) - 2x \cos\left(x^4\right)}{1 + x^2}.$$

Let $x_n = \left((4n + 1)\,\pi/2\right)^{1/4}$ so that $\cos\left(x_n^4\right) = 0$ while $\sin\left(x_n^4\right) = 1$. Then

$$f'\left(x_n\right) = \frac{4\left((4n + 1)\,\pi/2\right)^{3/4}}{1 + \left((4n + 1)\,\pi/2\right)^{1/2}}$$

and $\lim_{n \to \infty} f'(x_n) = +\infty$, so f' is unbounded. *Remark*: The key idea here is that f be the quotient of a highly oscillating but bounded function and a function that goes to infinity at infinity.

4. (p. 104, 5) We suppose that $f'(x) \neq 0$ on (a, b) and we wish to prove that f is monotone on (a, b). For any closed interval $[c, d]$ contained in (a, b), f is continuous on $[c, d]$ and differentiable on (c, d). We claim that f is monotone on $[c, d]$. If not, f has an interior minimum or an interior maximum at α, and it follows that $f'(\alpha) = 0$, contradicting the hypothesis. Hence, f is monotone on every subinterval $[c, d]$ of (a, b), and hence monotone on (a, b).

5. (p. 104, 6) Consider the function $g(x) = f(x) - cx$. We have $g'(x) = f'(x) - c$ so that $g'(a) < 0$ and $g'(b) > 0$. If $g'(x) \neq 0$ in (a, b), it follows from the previous problem that g is either strictly increasing or strictly decreasing on (a, b). This contradicts the fact that $g'(a)$ and $g'(b)$ have opposite signs. Thus, there is a point $\alpha \in (a, b)$ so that $g'(\alpha) = 0$, that is, $f'(\alpha) = c$.