Math 575
Problem Set #13 Solutions

1. (112, prob. 3) Suppose that f is nonnegative on $[a,b]$ and $\int_a^b f = 0$. Suppose there is an x_0 with $f(x_0) = c > 0$. Taking $\varepsilon = c/2$ there is a $\delta > 0$ so that $|f(y) - f(x)| < \varepsilon$ for $|x - y| < \delta$, so that $f(y) \geq c/2$ in $(x - \delta, x + \delta) \cap [a,b]$. Let P be a partition one of whose intervals is $[x - \delta, x + \delta] \cap [a,b]$. Then $L(P,f) \geq c\delta > 0$, contradicting the fact that $L(P,f) \leq \int_a^b f$. Hence, $f(x) = 0$ on $[a,b]$.

2. (112, prob. 4) Suppose that $g(1/n) = 1$ for $n \in \mathbb{N}$ and $g(x) = 0$ otherwise. We claim that $\int_0^1 g(x)\, dx$ exists. Observing that $L(P,f) = 0$ for any partition P, it suffices to show that, given $\varepsilon > 0$, there is a partition P with $U(P,f) < \varepsilon$. The idea is to isolate the discrete set $\{1/n\}_{n=1}^{\infty}$ in intervals of total size ε, and use the fact that $g(x)$ is identically zero on the complementary intervals. We'll suppose without loss of generality that $0 < \varepsilon < 1/2$. Each point $y_n = 1/n$ is contained in an interval $I_n = \left[y_n - \varepsilon 2^{-n-1}, y_n + \varepsilon^{2^{-n-1}}\right]$ which has total length $\varepsilon 2^{-n}$ (in the case $n = 1$, we take the interval to be $[1 - \varepsilon/2, 1]$. For the intervals to be nonoverlapping, we need
$$\frac{1}{n+1} + \varepsilon 2^{-n-2} < \frac{1}{n} - \varepsilon 2^{-n-1}$$
for all n, or
$$\varepsilon < \frac{2}{3} \frac{1}{n(n+1)} 2^{n+1}.$$
It is easy to see that the right-hand side is bounded below by 2/3 so that the condition $0 < \varepsilon < 1/2$ suffices. Now choose M so large that $1/M < \varepsilon/2$ and choose P as follows: P consists of the interval $[0, 1/M]$, the intervals $\{I_n\}_{n=1}^{M-1}$, and all of the complementary intervals where g is zero. Then
$$\begin{aligned} U(P,g) &= \frac{1}{M} + \sum_{n=1}^{M-1} 1 \cdot \varepsilon 2^{-n-1} \\ &< \varepsilon/2 + \varepsilon/2 \\ &< \varepsilon \end{aligned}$$
as claimed.

3. (116, prob. 2) Let $f(x) = \exp(-1/x)$. For $x > 0$ we have
$$\begin{aligned} f'(x) &= -x^{-2} e^{-1/x} \\ f''(x) &= \left(2x^{-3} + x^{-4}\right) e^{-1/x} \end{aligned}$$

and we claim that

$$f^{(n)}(x) = Q_{2n}(1/x)e^{-1/x}$$

where Q_{2n} is a polynomial of degree $2n$. The formulas above show that this holds for $n = 1$. Supposing that $f^{(n)}$ takes this form we have

$$f^{(n+1)}(x) = \left(Q'_{2n}(1/x) + Q_{2n}(1/x)\right)(-1/x^2)e^{-1/x}$$

so setting

$$Q_{2n+2}(z) = -z^2\left[Q'_{2n}(z) + Q_{2n}(z)\right]$$

we see that $f^{(n+1)}(x) = Q_{2n+2}(1/x)e^{-1/x}$ and $\deg Q_{2n+2} = \deg Q_{2n} + 2$. On the other hand, for $x < 0$, $f^{(n)}(x) = 0$ for all n. To compute the derivatives of f at zero and prove their continuity, we note that if P_n is any polynomial of degree n,

$$\lim_{x \downarrow 0} P_n(1/x)e^{-1/x} = \lim_{x \to \infty} P_n(z)e^{-z} = 0$$

by applying L'Hospital's rule to $\lim_{x\to\infty}\left(P_n(z)/e^z\right)$. We claim that $f^{(n)}(0) = 0$ and we will establish this inductively. It clearly holds for $n = 0$ and for $n \geq 1$ we consider separately the left- and right-hand limits of the difference quotient $\left(f^{(n)}(h) - f^{(n)}(0)\right)/h = f^{(n)}(h)/h$. The left-hand limit is trivial and the right-hand limit is

$$\begin{aligned}\lim_{h\downarrow 0}\frac{f^{(n)}(h)}{h} &= \lim_{h\downarrow 0}\frac{1}{h}Q_{2n}(1/h)e^{-1/h}\\ &= \lim_{h\downarrow 0}P_{2n+1}(1/h)e^{-1/h}\\ &= 0.\end{aligned}$$

It follows that the nth Taylor polynomial for $f(x)$ is the zero polynomial for all n.

4. (117, prob. 8) Consider the sequence of functions

$$f_n(x) = \begin{cases} n & x \in [0, 1/n] \\ n^2(2/n - x) & x \in [1/n, 2/n] \\ 0 & x \in [2/n, 1] \end{cases}.$$

Clearly, each f_n is continuous and $\int_0^1 f_n(x)dx = 3/2$. On the other hand, $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in (0, 1]$.

5. (117, prob. 11) Suppose $f \in C[a, b]$ and $\int_a^b x^n\ f(x)\ dx = 0$ for all non-negative integers n. We will suppose that $\|f\| > 0$ since otherwise there is nothing to prove. By the linearity of the integral, $\int P(x)f(x)\ dx = 0$ for any polynomial P. On the other hand, by the Weierstrass approximation theorem, for any $\varepsilon > 0$ there is a polynomial P so that

$$\|f - P\| < \frac{\varepsilon}{(b-a)\,\|f\|}$$

Since

$$\int_a^b f(x)^2 \ dx = \int_a^b f(x)P(x) \ dx + \int_a^b f(x)\left(P(x) - f(x)\right) dx$$
$$= \int_a^b f(x)\left(P(x) - f(x)\right) \ dx$$

we may estimate

$$\int_a^b f(x)^2 dx \le \int_a^b |f(x)|\,|P(x) - f(x)| \ dx$$
$$\le (b-a)\,\|f\|\,\|P - f\|$$
$$< \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\int_a^b f(x)^2 \ dx = 0$. From a previous exercise, it now follows that $f(x)^2 = 0$ on $[a, b]$, and hence f is the zero function.