Math 575
Fall 2018
Solutions to Problem Set # 3

(1) (p. 37, 1) (**3 points**) Suppose that $\{a_n\}$ is a positive, nondecreasing, convergent sequence and $\{b_n\}$ is a complex sequence with the property that $|b_{n+1} - b_n| \leq a_{n+1} - a_n$. We claim that $\{b_n\}$ also converges. We will use the Cauchy criterion. We estimate

$$\begin{aligned} |b_{n+m} - b_n| &\leq \sum_{k=0}^{m-1} |b_{n+k+1} - b_{n+k}| \\ &\leq \sum_{k=0}^{m-1} (a_{n+k+1} - a_{n+k}) \\ &= a_{n+m} - a_n \\ &= |a_{n+m} - a_n| \end{aligned}$$

where in the first step we used the triangle inequlity, in the second step we used the hypothesis, in the third step we used the fact that the sum telescopes, and in the last step we used the fact that $\{a_n\}$ is nondecreasing.

This is an example of concise mathematical writing to convey the idea behind a chain of inequalities

Now let $\varepsilon > 0$ be given. Since $\{a_n\}$ converges, there is an $N \in \mathbb{N}$ so that $|a_{n+m} - a_n| < \varepsilon$ for all $n \geq N$ and $m \geq 1$. By the inequality just proved, the same is true of $|b_{n+m} - b_n|$. By the Cauchy criterion, $\{b_n\}$ converges.

(2) (p. 38, 3) (**3 points**) Suppose that $\{z_n\}$ is a convergent sequence of complex numbers and let

$$w_n = \frac{z_1 + z_2 + \ldots + z_n}{n}$$

be the sequence of arithmetic means. Because $\{z_n\}$ is convergent, it is bounded, so there is a positive number C so that $|z_n| \leq C$ for all n. By increasing C if needed, we may suppose that $|z| \leq C$ as well, so that $|z_n - z| \leq 2C$ by the triangle inequality.

Here I introduce a piece of notation (C) to be used later, to avoid having to explain it in mid-proof

We claim that

$$\lim_{n\to\infty} w_n = z.$$

To prove this, note that we can compute

$$\begin{aligned} |w_n - z| &\leq \left| \sum_{k=1}^{n} \frac{w_k - z}{n} \right| \\ &\leq \sum_{k=1}^{n} \frac{|z_k - z|}{n}. \end{aligned}$$

The intuition is that, for large k, $|z_k - z|$ is small, so the average of such terms will be small. The rest should be controlled by averaging over a large enough number of terms. To make this idea precise, let $\varepsilon > 0$ be given and choose N_1 so that $|z_k - z| < \varepsilon/2$ for $k \geq N_1$. We may then estimate

$$\begin{aligned} |w_n - z| &\leq \sum_{k=1}^{N_1} \frac{|z_k - z|}{n} + \sum_{k=N_1+1}^{n} \frac{|z_k - z|}{n} \\ &\leq \frac{2CN_1}{n} + \frac{n - N_1}{n}\frac{\varepsilon}{2} \\ &\leq \frac{2CN_1}{n} + \frac{\varepsilon}{2} \end{aligned}$$

where C is the constant chosen above.

This is where C gets used-note the reminder to the reader that C has already been defined

Since N_1 is now chosen, we may choose N_2 so that $2CN_1/n < \varepsilon/2$ for all $n \geq N_2$. It then follows that, for all $n \geq N_2$,

$$|w_n - z| < \varepsilon$$

as was to be proved.

(3) (p. 40, 1) (**4 points**)

(a) (**2 points**) Let $a = \liminf_{n\to\infty} x_n$ and $b = \limsup_{n\to\infty} x_n$. GIven any $\varepsilon > 0$ we can find an N so that $a - \varepsilon < x_n < b + \varepsilon$ for all $n \geq N$. Thus, if $\{y_n\}$ is any convergent subsequence with $y = \lim_{n\to\infty} y_n$, we can insure (by increasing N if needed) that $y_n - \varepsilon < y < y_n + \varepsilon$ and $a - \varepsilon < y_n < b + \varepsilon$ for all $n \geq N$. It follows that $a - 2\varepsilon < y < b + 2\varepsilon$ for any $\varepsilon > 0$, and hence $a \leq y \leq b$.

(b) (**2 points**) Recall that if

$$\begin{aligned} a_m &= \inf_{n\geq m} x_n, \\ b_m &= \inf_{n\geq m} y_n, \end{aligned}$$

$a_m \nearrow a$ is shorthand for "a_m is monotone nondecreasing with limit a." I'll bet you can guess what $b_m \searrow b$ means!

then $a_m \nearrow a$ and $b_n \searrow b$. We'll find a subsequence of $\{x_n\}$ that converges to a; the construction of a subsequence converging to b is similar. For any integer k we can find an m_k so that $a - \frac{1}{k} \leq a_m \leq a$. By the definition of a_m there is an x_{n_k} with $n_k \geq m_k$ so that $|x_{n_k} - a_{m_k}| < \frac{1}{k}$. Combining these inequalities we see that

$$a - \frac{2}{k} < x_{n_k} < a + \frac{1}{k}$$

for each $k \in \mathbb{N}$. Thus, let $\varepsilon > 0$ be given and choose K so that $\frac{2}{k} < \varepsilon$. For any $k \geq K$ we have $|x_{n_k} - a| < (2/k) < \varepsilon$. This shows that $x_{n_k} \to a$.