Math 575
Fall 2018
Solutions to Problem Set # 6

(1) (p. 74, 10) (8 points)

(a) (6 points)

To show that d_1 is a metric on $\mathbb{R}^n$:

Each of (i)–(iii) is worth 1 point

(i) For any $x \in \mathbb{R}^n$,

$$d_1(x, x) = \sum_{j=1}^{n} |x_i - x_i| = 0$$

(ii) For any $x, y \in \mathbb{R}^n$,

$$d_1(x, y) = \sum_{j=1}^{n} |x_i - y_i| = \sum_{j=1}^{n} |y_i - x_i| = d_1(y, x).$$

(iii) For any points $x, y, z \in \mathbb{R}^n$,

$$\begin{aligned} d_1(x, z) &= \sum_{j=1}^{n} |x_i - z_i| \\ &\leq \sum_{j=1}^{n} (|x_i - y_i| + |y_i - z_i|) \\ &= \sum_{j=1}^{n} |x_i - y_i| + \sum_{j=1}^{n} |y_i - z_i| \\ &= d_1(x, y) + d_1(y, z). \end{aligned}$$

To show that d_∞ is a metric on $\mathbb{R}^n$:

Each of (i)–(iii) is worth 1 point

(i) For any $x \in \mathbb{R}^n$,

$$d_\infty(x, x) = \sup\{|x_i - x_i|, 1 \leq i \leq n\} = 0$$

(ii) For any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} d_\infty(x, y) &= \sup\{|x_i - y_i|, 1 \leq i \leq n\} \\ &= \sup\{|y_i - x_i| : 1 \leq j \leq n\} \\ &= d_\infty(y, x). \end{aligned}$$

(iii) For any points $x, y, z \in \mathbb{R}^n$,

$$\begin{aligned} d_\infty(x, z) &= \sup\{|x_i - z_i|, 1 \leq i \leq n\} \\ &\leq \sup\{|x_i - y_i| + |y_i - z_i|, 1 \leq j \leq n\} \\ &= \sup\{|x_i - y_i|, 1 \leq i \leq n\} \\ &\quad + \sup\{|y_i - z_i|, 1 \leq i \leq n\} \\ &= d_\infty(x, y) + d_\infty(y, z). \end{aligned}$$

(b) (2 points) It is easy to see that, for any $1 \leq i \neq n$, $|x_i - y_i| \leq d_1(x, y)$, so that

$$d_\infty(x, y) \leq d_1(x, y).$$

On the other hand

$$d_1(x, y) \leq \sqrt{n} \sup\{|x_i - y_i|, 1 \leq i \leq n\} = \sqrt{n} d_\infty(x, y).$$

This shows that

$$\left(\sqrt{n}\right)^{-1} d_\infty(x, y) \leq d_1(x, y) \leq \sqrt{n} d_\infty(x, y)$$

which shows that the two norms on $\mathbb{R}^n$ are equivalent.

(2) (p. 74, 3) (2 points) Suppose that V is a vector space with norm $\|\cdot\|$, i.e.,

(i) $\|x\| \geq 0$ for all $x \in V$,
(ii) $\|\lambda x\| = |\lambda| \|x\|$ for any scalar λ and $x \in V$, and
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

V can either be a real or a complex vector space, so the λ here is either an element of $\mathbb{R}$ or $\mathbb{C}$. The property $\|\lambda x\| = |\lambda| \|x\|$ implies that $\|\mathbf{0}\| = 0$.

Then, letting $d(x, y) = \|x - y\|$,

(i) $d(x, x) = \|x - x\| = \|\mathbf{0}\| = 0$,
(ii) $d(x, y) = \|x - y\| = \|(-1)(y - x)\| = \|y - x\| = d(y, x)$
(iii) $d(x, z) = \|x - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$