Math 575
Fall 2018
Solutions to Problem Set # 7

(1) (p. 78, 2) (3 points) First, suppose that p is a limit point of B. There is a $q_1 \in N_1(p)$ with $p \neq q$ and $q \in B$. Let $r_1 = d(p, q_1)/2$. There is a $q_2 \in N_{r_1}(p)$ with $q_2 \in B$ and $q_2 \neq p$. At the kth step, given $q_k \in B$ with $q_k \neq p$, let $r_k = d(p, q_k)/2$ and choose $q_{k+1} \in N_{r_k}(p)$, $q_{k+1} \neq p$. Continuing in this way we obtain a sequence of distinct points q_1, q_2, … with the properties that $q_k \in B$, $q_k \neq p$, and q_k is distinct from the points q_j with $1 \leq j < k$, and a strictly decreasing sequence r_1, r_2, … with the properties that $r_k \leq 2^{-k} r_1$ and $p_n \in N_{r_N}(p)$ for all $n \geq N$. Given $\varepsilon > 0$ choose N so that $r_N < \varepsilon$. Then $d(q_n, p) < \varepsilon$ for all $n \geq N$, so $q_n \to p$ as $n \to \infty$.

(2) (p. 81, 1)

(a) (2 points) Suppose that A is a finite subset of a metric space S, and let $\mathcal{U}$ be an open cover of A. Denoting by $\{p_k\}_{k-1}^{N}$ the points of A, the cover $\{U_k\}_{k=1}^{N}$, where $p_k \in U_k$, is a finite subcover of A. Hence A is compact.

(b) (2 points) Suppose that S has the discrete metric and that A is a compact subset of S. Let $\mathcal{U}$ be a cover of A by neighborhoods of the form $N_{1/2}(p)$ for $p \in A$. The set $N_{1/2}(p)$ is an open neighborhood which contains only p. Since A is compact, the cover $\mathcal{U}$ contains a finite subcover each of whose open sets contain exactly one element of A. Hence, S is finite.

(c) (1 point) Let S be a countable set with the discrete metric

$$d(p,q) = \begin{cases} 0 & p = q \\ 1 & p \neq q \end{cases}.$$

The set S is itself bounded since, for any $p, q \in S$, $d(p,q) \leq 1$. On the other hand, since S is countably infinite, S cannot be compact.

(3) (p. 81,2) (2 points) Suppose that A and B are compact sets of a metric space S, and let $C = A \cup B$. We claim that C is compact. Let $\mathcal{U}$ be any open cover of C. Any cover of C is also a cover of A and hence has a finite subcover $\{U_j\}_{j=1}^{N}$ of A. Such a cover also covers B so that there is a finite subcover $\{V_j\}_{j=1}^{M}$ of B. Hence, $\{U_j\} \cup \{V_k\}$ is a finite subcover of $A \cup B$.