Math 575
Fall 2018
Solutions to Problem Set # 9

(1) (p. 89, # 1) (2 points) Suppose that $f : S \to T$ is continuous at p and that $\{p_n\}$ is a sequence from S with $\lim_{n\to\infty} p_n = p$. We will show that $f(p_n) \to f(p)$ in T. Let $\varepsilon > 0$ be given. There is a $\delta > 0$ so that $d_T(f(p), f(q)) < \varepsilon$ whenever $d_S(p, q) < \delta$. Now choose N so that $d_S(p, p_n) < \delta$ whenever $n \geq N$. It now follows that $d_T(f(p), f(p_n)) < \varepsilon$ for all $n \geq N$, which shows that $f(p_n) \to f(p)$.

(2) (p. 89, # 3) (4 points) Suppose that $f : S \to T$, $g : T \to U$, that f is continuous at p, and that g is continuous at $f(p)$. We wish to show that $g \circ f$ is continuous at p. Let $z = f(p)$ and let $\varepsilon > 0$ be given. There is a $\delta_1 > 0$ so that $d_U((g(z), g(t)) < \varepsilon$ whenever $d_T(z, t) < \delta_1$. There is a $\delta_2 > 0$ so that $d_T(f(p), f(q)) < \delta_1$ provided $d_S(p, q) < \delta_2$. Since $z = f(p)$, it now follows that $d_U((g \circ f)(p), (g \circ f)(q)) < \varepsilon$ whenever $d_S(p, q) < \delta_2$. Hence, $g \circ f$ is continuous at p.

(3) (p. 89, # 4) (4 points) Let $\varepsilon > 0$ be given. We wish to show that the map $p \mapsto d(a, p)$ is uniformly continuous. If p and q are any points of S,

$$d(a, p) \leq d(a, q) + d(q, p), \quad d(a, q) \leq d(a, p) + d(p, q)$$

so that

$$d(a, p) - d(a, q) \leq d(q, p), \quad d(a, q) - d(a, p) \leq d(p, q)$$

from which we deduce

$$|d(a, p) - d(a, q)| \leq d(p, q).$$

Given $\varepsilon > 0$, we can choose $\delta = \varepsilon$ to insure that $|d(a, p) - d(a, q)| < \varepsilon$ whenever $d(p, q) < \delta$. Since the choice of δ is independent of p and q, the map $p \mapsto d(a, p)$ is uniformly continuous.

(4) (p. 89, # 6) (not graded) Here is one example due to a student from another year (!) when I taught this course. Since B is not a closed subset of S, there must be at least one limit point q of B that is not contained in B and define

$$f(p) = \frac{d(p, q)}{1 + d(p, q)}.$$

Since $0 \leq d(p, q)$ and the function $g(t) = t/(t + 1)$ satisfies $0 \leq t < 1$ for $t \geq 0$, it follows that $0 \leq f(p) < 1$ so f is bounded.

The function f is also uniformly continuous. This is a consequence of the estimate

$$\left|\frac{t}{1+t} - \frac{s}{1+s}\right| = \left|\frac{t-s}{(1+t)(1+s)}\right| \leq |t - s|$$

Because q is a limit point of B, there is a sequence $\{p_n\}$ from B so that $d(p_n, q) \to 0$ as $n \to \infty$, hence $f(p_n) \to 0$ as $n \to \infty$. From these facts, it follows that the greatest lower bound of the set $\{f(p) : p \in B\}$ is zero. However, since $q \notin B$, it follows that $d(p, q) > 0$ for every $p \in B$. Hence, f does not assume a minimum value.