MATH 676
PRODUCT MEASURE

1. Tonelli's Theorem

Theorem 1. *Suppose $f(x,y)$ is a nonnegative measurable function on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then:*

The proof is Fubini's theorem and monotone convergence

(i) *For almost every $y \in \mathbb{R}^{d_2}$, the slice function f^y is measurable on $\mathbb{R}^{d_1}$.*
(ii) *The function $g(y) = \int_{R^{d_1}} f^y(x)\,dx$ is measurable on $\mathbb{R}^{d_2}$.*
(iii) *The formula*

$$\int_{R^{d_2}} \left(\int_{R^{d_1}} f(x,y)\,dx \right) dy = \int_{\mathbb{R}^d} f(x,y)\,dx\,dy$$

holds in the extended sense.

Corollary 2. *If E is a measurable set in $R^{d_1} \times \mathbb{R}^{d_2}$, then for almost every $y \in \mathbb{R}^{d_2}$ the slice E^y is a measurable subset of $\mathbb{R}^{d_1}$. Moreover, $m(E^y)$ is a measurable function of y and*

An immediate consequence of Theorem 1 applied to χ_E

$$m(E) = \int_{R^{d_2}} m(E^y)\,dy.$$

Note that there are non-measurable sets on $\mathbb{R} \times \mathbb{R}$ for which all of the slices E_x and E^y are measurable!

2. Product Measure

On the other hand:

Proposition 3. *Suppose that $E_1 \subset \mathbb{R}^{d_1}$ and $E_2 \subset \mathbb{R}^{d_2}$ are measurable. Then $E = E_1 \times E_2$ is a measurable subset of $\mathbb{R}^{d_1+d_2}$ and $m(E) = m(E_1)m(E_2)$ with the understanding that, if one of the sets has measure zero, then $m(E) = 0$.*[1]

Proof. (Sketch) If E is measurable then the formula for $m(E)$ is an application of Theorem 1 to $\chi_E(x,y) = \chi_{E_1}(x)\chi_{E_2}(y)$. The measurability follows from approximating E_1 and E_2 by G_δ sets and using the fact that the Cartesian product of open sets is open. □

Corollary 4. *Suppose that f is a measurable function on $\mathbb{R}^{d_1}$. Then the function $F(x,y) = f(x)$ on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ is measurable.*

Proof. (Sketch) The set $\{F < a\}$ is $\{f < a\} \times \mathbb{R}^{d_2}$, so appeal to Proposition 3. □

The next result connects Lebesgue integration to the "area under the graph."

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[1]This covers the possibility that one of E_1 or E_2 has infinite measure and the other has measure zero.

Corollary 5. *Suppose that f is a nonnegative function on $\mathbb{R}^d$, and let*

$$\mathcal{A} = \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leq y \leq f(x)\right\}.$$

Then:

(i) The function f is measurable on $\mathbb{R}^d$ if and only if the set $\mathcal{A}$ is measurable in $\mathbb{R}^{d+1}$.

(ii) If either f or $\mathcal{A}$ is measurable, then

$$\int_{\mathbb{R}^d} f(x)\, dx = m(\mathcal{A}).$$

Proof. (i) If f is measurable on $\mathbb{R}^d$, then $F(x, y) = y - f(x)$ is measurable on $\mathbb{R}^{d+1}$ by Corollary 4 (applied twice). Since

$$\mathcal{A} = \{F \leq 0\} \cap \{y \geq 0\}$$

it follows that $\mathcal{A}$ is measurable. Conversely, if $\mathcal{A}$ is measurable, note that $\mathcal{A}_x = [0, f(x)]$. Hence, by Theorem 1, $m(\mathcal{A}_x) = f(x)$ is measurable. By the same Theorem, we may compute

$$m(\mathcal{A}) = \int \chi_{\mathcal{A}(x,y)}\, dx\, dy = \int_{\mathbb{R}^d} m(\mathcal{A}_x)\, dx = \int_{\mathbb{R}^d} f(x)\, dx.$$

□

The last result will pave the way for a discussion about the *convolution* of two functions f and g, namely

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y) g(y)\, dy.$$

Proposition 6. *If f is a measurable function on $\mathbb{R}^d$, then the function $F(x, y) = f(x - y)$ is measurable on $\mathbb{R}^d \times \mathbb{R}^d$.*

Proof. For a subset E of $\mathbb{R}^d$, let

$$\widetilde{E} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x - y \in \mathbb{R}^d\}.$$

It will suffice to show that if E is measurable in $\mathbb{R}^d$, then $\widetilde{E}$ is measurable in $\mathbb{R}^d \times \mathbb{R}^d$.

First, if O is open, then $\widetilde{O}$ is open, and hence measurable.

Next, if E is a G_δ set, then $\widetilde{E}$ is a G_δ set, hence measurable.

Now suppose that E is a set of measure zero in $\mathbb{R}^d$. There is a sequence $\mathcal{O}_n$ of open sets with $m(\mathcal{O}_n) \to 0$ and $E \subset \mathcal{O}_n$ for each n. Let $B_k = \{y \in \mathbb{R}^d : |y| < k\}$. For each n and k

$$\begin{aligned} m(\widetilde{O_n} \cap B_k) &= \int \chi_{\mathcal{O}_n}(x - y) \chi_{B_k}(y)\, dy\, dx \\ &= \int \left(\int \chi_{\mathcal{O}_n}(x - y)\, dx \right) \chi_{B_k}(y)\, dy \\ &= m(\mathcal{O}_n) m(B_k). \end{aligned}$$

Thus if $\widetilde{E_k} = \widetilde{E} \cap B_k$, we can compute that $m(\widetilde{E_k}) = 0$ by taking $n \to \infty$. Since $\widetilde{E_k} \nearrow \widetilde{E}$, it follows that $m(\widetilde{E}) = 0$.

Since any measurable set can be written $E = G - Z$ for G a G_δ set and Z a set of measure zero, it follows that E is measurable.

□