Math 676
Final Exam

Your Name: ______________________________

Instructions: This is a two-hour, closed-book exam. You must answer all five of the problems in the space provided.

Problem	1	2	3	4	5	Total
Possible	20	20	20	20	20	100
Score						

1. (20 points) A real-valued function f on $\mathbb{R}$ is said to be *measurable* if the sets $E_\alpha = \{x \in \mathbb{R} : f(x) > \alpha\}$ are measurable sets for each $\alpha \in \mathbb{R}$. Suppose that $\{f_k\}_{k=1}^\infty$ is a monotone nondecreasing sequence of measurable functions with $\lim_{k\to\infty} f_k(x) = f(x)$ for every $x \in \mathbb{R}$. Using the definition of measurability, show directly that f is a measurable function.

Solution:

Fix $\alpha \in \mathbb{R}$ and let

$$E_k = \{x \in \mathbb{R} : f_k(x) > \alpha\}\,.$$

Denote

$$E = \{x \in \mathbb{R} : f(x) > \alpha\}\,.$$

Since $f_k(x) \leq f_{k+1}(x)$ for each x, it follows that any $x \in E_k$ belongs to E_{k+1}. Thus, $\{E_k\}$ is an increasing sequence of measurable sets.

We claim that $E = \cup_{k=1}^\infty E_k$. If $x \in E$, then there is a k so that $f_k(x) > \alpha$. Hence $E \subset \cup_{k=1}^\infty E_k$. On the other hand, if $x \in \cup_{k=1}^\infty E_k$, $x \in E_k$ for at least one k, and hence $f(x) \geq f_k(x) > \alpha$.

Since each E_k is measurable, E, as a countable union of measurable sets, is measurable.

2. (20 points) The Fourier transform $\widehat{f}$ of a function $f \in L^1(\mathbb{R})$ is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)\, dx$$

(a) (5 points) Show that $\widehat{f}$ is a bounded, continuous function.

Solution: Since $f \in L^1(\mathbb{R})$, it follows from the triangle inequality that

$$\left|\widehat{f}(\xi)\right| \leq \int |f(x)|\, dx = \|f\|_{L^1}.$$

Since $\left|e^{-2\pi i x\xi} f(x)\right| \leq |f(x)|$ and $\lim_{h\to 0} e^{2\pi i(x+h)\xi} f(x) = e^{2\pi i x\xi} f(x)$ for almost every x (we need finiteness of $f(x)$ which is true a.e. since f is integrable, the continuity follows by the Dominated Convergence Theorem.[1]

(b) (5 points) Show that if $\chi_{[a,b]}(x)$ is the characteristic function of $[a, b]$, then

$$\int e^{2\pi i x\xi} \chi_{[a,b]}(x)\, dx = \frac{e^{2\pi i b\xi} - e^{2\pi i a\xi}}{2\pi i\xi}.$$

You may assume that the complex-valued function $e^{2\pi i c x}$ has antiderivative $\dfrac{e^{2\pi i c x}}{2\pi i c}$.

Solution: A direct computation.

(c) (10 points) Show that $\widehat{f}(\xi) \to 0$ as $\xi \to \infty$. You may assume that linear combinations of characteristic functions of closed intervals are dense in $L^1(\mathbb{R})$.

Solution: From the formula above we see that

$$\lim_{|\xi| \to \infty} \widehat{\chi_{[a,b]}}(\xi) = 0.$$

If $f \in L^1$, for any $\varepsilon > 0$ we may approximate f as a finite linear combination $\sum_{k=1}^{N} c_k \chi_k$ where χ is the characteristic function of an interval and

$$\left\| f - \sum_{k=1}^{N} c_k \chi_k \right\|_{L^1} < \varepsilon/2.$$

On the other hand, we may find R so that

$$\sum_{k=1}^{N} |c_k| \left| \widehat{\chi_{[a,b]}}(\xi) \right| < \varepsilon/2$$

for $|\xi| \geq R$. Hence

$$\left| \widehat{f} \right| \leq \left\| f - \sum_{k} c_k \chi_k \right\|_{L^1} + \sum_{k=1}^{N} |c_k| \left| \widehat{\chi_{[a,b]}}(\xi) \right| < \varepsilon.$$

3. (20 points) Suppose that $f(x,y)$ is a measurable function on $[0,1]\times[0,1]$, that $f(x,y)$ is an integrable function of y for each x, and that $(\partial f/\partial x)(x,y)$ is a bounded function of (x,y) for $(x,y)\in(0,1)\times(0,1)$

   (a) (10 points) Show that $\partial f/\partial y$ is a measurable function of y for each x. In this problem, you may assume that the pointwise limit of a sequence of measurable functions is measurable.

Solution: Fix $x\in[0,1]$, fix a sequence $\{h_n\}$ with $h_n\to 0$, and let $g_n(y) = h_n^{-1}\left[f(x+h_n,y)-f(x,y)\right]$.[2] Each function $g_n(y)$ is measurable since $f(x,\cdot)$ is integrable, hence measurable, for each x. Since f is differentiable with respect to y we have $g_n(y)\to f_x(y)$ as $n\to\infty$. Hence $f_x(y)$, as a pointwise limit of a sequence of measurable functions, is measurable.

Alternatively, one can note that the function

$$F_n(x,y) = h_n^{-1}\left[f(x+h_n,y)-f(x,y)\right]$$

is measurable on $[0,1]\times[0,1]$ and converges pointwise almost everywhere as $n\to\infty$ to a bounded measurable function on $[0,1]\times[0,1]$, and hence $\partial f/\partial x$ is a bounded measurable function on $[0,1]\times[0,1]$. It then follows from Tonnelli's Theorem that $(\partial f/\partial x)(x,\cdot)$ is measurable on $[0,1]$ for almost every x.

   (b) (10 points) Show that

$$\frac{\partial}{\partial x}\left(\int_0^1 f(x,y)\,dy\right) = \int_0^1 \frac{\partial f}{\partial x}(x,y)\,dy$$

Solution: Let

$$G(x) = \int_0^1 f(x,y)\,dy.$$

We seek to show that G is differentiable and

$$G'(x) = \int_0^1 \frac{\partial f}{\partial x}(x,y)\,dy.$$

Consider the difference quotient (with $h=1/n$)

$$\frac{G(x+h)-G(x)}{h} = \int_0^1 \frac{f(x+1/n,y)-f(x,y)}{1/n}\,dy.$$

For each fixed x,

$$F_n(y) = \frac{f(x+1/n,y)-f(x,y)}{1/n}$$

converge pointwise to the bounded function $(\partial f/\partial x)(x,y)$. Moreover, by the Mean Value Theorem, for each n there is a c_n dependening on n and y so that

$$\left|\frac{f(x+1/n,y)-f(x,y)}{1/n}\right| = |f_x(c_n,y)| \leq M$$

where M is a constant that bounds $f_x(x,y)$ for $(x,y) \in (0,1)\times(0,1)$. Since the integrand converges pointwise to $(\partial f/\partial x)(x,y)$ and the approximants are uniformly bounded, it follows from the Bounded Convergence Theorem that the desired equality holds.

4. (20 points) Suppose that f is absolutely continuous and Lebesgue integrable on $\mathbb{R}$. Show that $\lim_{|x|\to\infty} f(x) = 0$.

Solution: Suppose that $\limsup_{|x|\to\infty} |f(x)| = c > 0$. There is a sequence of points $\{x_n\}$ with $x_n \to \infty$ as $n \to \infty$ so that $|f(x_n)| \geq c/2$. By passing to a subsequence if needed, we may assume that $|x_{n+1} - x_n| > 1$. Since f is absolutely continuous, there is a $\delta > 0$ so that $|f(x) - f(y)| < c/4$ if $|x - y| < \delta$. Hence $|f(y)| > c/4$ on an infinite set of disjoint intervals $(x_n - \delta, x_n + \delta)$, which contradicts the fact that f is Lebesgue integrable.

5. (20 points) For both parts of this problem, suppose that f is an absolutely continuous function on $[a, b]$.

   (a) (10 points) Show that f maps sets of measure zero to sets of measure zero.

   Solution: Suppose that Z is a set of measure zero in $[a, b]$ and let $\varepsilon > 0$ be given. There is a $\delta > 0$ so that for any finite collection of disjoint intervals $\{(c_k, d_k)\}$, $\sum_k |F(d_k) - F(c_k)| < \varepsilon$ whenever $\sum_k |d_k - c_k| < \delta$. We can find an open set $\mathcal{O}$ so that $Z \subset \mathcal{O}$ and $m(\mathcal{O}) < \delta$. The set $\mathcal{O}$ is a disjoint union of open intervals $I_k = (a_k, b_k)$, and hence $\sum_{k=1}^{\infty} |b_k - a_k| < \delta$. Because f is continuous on $[a_k, b_k]$, f achieves absolute extrema at points c_k and d_k in $[a_k, b_k]$. Assume that $c_k < d_k$. .Then $F(I_k)$ is an interval of length at most $|F(c_k) - F(d_k)|$. For any N, $\sum_{k=1}^{N} |F(c_k) - F(d_k)| < \varepsilon$ since $\sum_{k=1}^{N} |d_k - c_k| \leq \sum_{k=1}^{N} |b_k - a_k < \delta$. It now follows that $\sum_{k=1}^{\infty} |F(d_k) - F(c_k)| \leq \varepsilon$, so the measure of $f(\mathcal{O})$ is at most ε. Since $\varepsilon > 0$ is arbitrary, it follows that $f(Z)$ has measure zero.

   (b) (10 points) Using the result of part (a), show that f maps measurable sets to measurable sets.

   Solution: Any measurable set $E \subset [a, b]$ is the union of a set of measure zero and an F_σ set, i.e., a countable intersection of closed sets. Thus it suffices to show that f takes F_σ sets to F_σ sets. Any closed subset C of $[a, b]$ is compact, and hence $f(C)$ is compact since f is continuous. It follows that if $F = \cap_{n=1}^{\infty} C_n$, then $f(F) = \cap_{n=1}^{\infty} f(C_n)$ is an F_σ set. Hence, f maps measurable sets into measurable sets.