PROBLEM SET #3 SOLUTIONS

1. (**4 points**) Stein and Shakarchi, page 42, Exercise 20.

   (a) (**2 points**) The Cantor set $\mathcal{C}$ consists of all those x with ternary expansions of the form

   $$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}, \quad a_n \in \{0, 2\}$$

   so that $\mathcal{C}/2$ consists of all those x with ternrary expansions of the same form but with $a_n \in \{0, 1\}$. It is clear that any $x \in [0, 1]$ may be written as a sum of $x \in \mathcal{C}$ and $y \in \mathcal{C}/2$, so $\mathcal{C} + \mathcal{C}/2 \supset [0, 1]$ even though $m(\mathcal{C}) = 0$ and $m(\mathcal{C}/2) = 0$ since any cover of $\mathcal{C}$ by open cubes can be recaled to give a cover of $\mathcal{C}/2$ having half the measure.

   (b) (**2 points**) If $A = I \times \{0\} \subset \mathbb{R}^2$ and $B = \{0\} \times I \subset \mathbb{R}^2$, then $A + B = I \times I$, the unit square in $\mathbb{R}^2$. It is easy to see that $m(I \times \{0\}) = m(\{0\} \times I) = 0$ (cover I with small intervals and then extend these to small cubes) but clearly $m(I \times I) = 1$.

2. (**6 points**) Stein and Shakarchi, page 43, Exercise 23.

   Suppose that $f(x, y)$ is a function on $\mathbb{R}^2$ and is separately continuous in x and y. We wish to show that f is also measurable. It suffices to exhibit a sequence of measurable functions that converges to f since, by Property 3 of measurable functions, the pointwise limit of a sequence of measurable functions is measurable. For each positive integer n, define a piecewise linear function f_n as follows. For each integer ℓ and $x \in [\ell/n, (\ell+1)/n)$ define

   $$f_n(x, y) = f(\ell/n, y) + (nx - \ell) \cdot [f((\ell+1)/n, y) - f(\ell/n, y)].$$

   Since f_n is piecewise linear in x and each of the functions $y \mapsto f(\ell/n, y)$ and $y \mapsto f((\ell+1)/n, y)$ is a continuous function of y, it follows that $f_n(x, y)$ is jointly continuous in x and y, hence measurable. It remains to show that $f_n(x, y) \to f(x, y)$ as $x \to \infty$. If $x \in [\ell/n, (\ell+1)/n]$ then

   $$\begin{aligned} f_n(x, y) - f(x, y) &= (1 - \theta)\,[f(\ell/n, y) - f(x, y)] \\ &\quad + \theta\,[f((\ell+1)/n, y) - f(x, y)] \end{aligned}$$

where $\theta = nx - \ell \in [0,1]$. Since $x \mapsto f(x,y)$ is continuous for each fixed y, given $\varepsilon > 0$ we can find $\delta > 0$ so that $|f(x',y) - f(x,y)| < \varepsilon$ whenever $|x - x'| < \delta$. Given such a δ choose n so that $n < \delta$. It then follows that $|f_n(x,y) - f(x,y)| < \varepsilon$ for such n, showing that $f_n(x,y) \to f(x,y)$.

3. (**Not graded**) Stein and Shakarchi, page 43, Exercise 26.

Suppose that A and B are measurable sets of finite measure, that $A \subset E \subset B$, and that $m(A) = m(B)$. We claim that E is measurable. From the containment $E - A \subset B - A$ we see that the set $E - A$ is contained in a set of measure zero since $m(B) = m(A) + m(B - A)$. It follows that $Z = E - A$ has outer measure zero, hence is a measurable set of measure zero. Hence, $E = A \cup Z$, the union of measurable sets, is also measurable.