## PROBLEM SET #4 SOLUTIONS

1. (**Not graded**) Stein and Shakarchi, page 44, Exercise 28.

Suppose that $\mathcal{O}$ is an open set containing $E$ with $m_*(E) \geq \alpha m_*(\mathcal{O})$ for some $\alpha \in (0,1)$. We can write $\mathcal{O} = \cup_{n=1}^{\infty} I_n$ where the $I_n$ are disjoint open intervals by Theorem 1.3 in the text. We claim that $m_*(E \cap I) \geq \alpha m_*(I)$ for at least one of the $I_n$. If not, then $m_*(E \cap I_n) < \alpha m_*(I_n)$ for every $n$. We then have

$$\begin{aligned} m_*(E) &= m_* \left(\cup_{n=1}^{\infty} E \cap I_n\right) \\ &\leq \sum_{n=1}^{\infty} m_*(E \cap I_n) \\ &\leq \sum_{n=1}^{\infty} \alpha m_*(I_n) \\ &= \alpha m_*(\mathcal{O}), \end{aligned} \tag{1}$$

a contradiction.

It's worth commenting on why the last step in (1) is correct. If $\mathcal{O} = \cup I_n$ then $m_*(\mathcal{O}) \leq \sum_{n=1}^{\infty} m_*(I_n)$ by countable subbadditivity of outer measure. On the other hand, if $\{I_n\}_{n=1}^{k}$ is any finite set of disjoint open intervals, the finite collection $\{J_n\}_{n=1}^{k}$, where $J_n$ is a disjoint collection of closed intervals with $J_n \subset I_n$, satisfies $m_*(\cup_{n=1}^{k} J_n) = \sum_{n=1}^{k} m_*(J_n) \leq m_* \left(\cup_{n=1}^{k} I_n\right)$. We can approximate $m_*(I_n)$ by $m_*(J_n)$ with arbitrary precision and conclude that

$$m_* \left(\cup_{n=1}^{k} I_n\right) = \sum_{n=1}^{j} m_*(I_n).$$

It follows that

$$\sum_{n=1}^{k} m_*(I_n) \leq m_*(\mathcal{O})$$

for any $k$, Hence, finally (!), $m_*(\mathcal{O}) = \sum_{n=1}^{\infty} m_*(I_n)$.

2. (**4 points)** Stein and Shakarchi, page 44, Exercise 29

Suppose $E$ is a measurable subset of $\mathbb{R}$ having nonzero finite measure and consider the difference set

$$F = \{z \in R : z = x - y \text{ for some } x, y \in E\}.$$

Following the hint, there is an interval $I$ so that $m(E_0) > (9/10)m(I)$ where $E_0 = E \cap I$. Suppose that the difference set $F_0$ of $E_0$ (which is contained in $F$) contains no open interval. Since 0 lies in the difference set, then there is some number $a > 0$ so that $F_0 \setminus \{0\} \cap (-2a, 2a) = 0$. Te sets $E_0$ and $E_0 + a$ are disjoint since any $x$ in the intersection takes the form $y + a$ for another element of $E_0$, which is impossible. We then compute, on the one hand

$$m\left(E_0 \cup E_0 + a\right) = 2m(E_0).$$

But, on the other hand, $E_0 \cup (E_0 + a) \subset I \cup (I + a)$, so that

$$m(E_0 \cup E_0 + a) \leq m(I \cup (I + 2a)) < (1 + \varepsilon)m(I)$$

which contradicts the fact that $m(E_0) > (9/10)m(I)$.

3. (**Not graded**) Stein and Shakarchi, page 45, Exercise 37.

Remember this problem from our first in-class exercise?

Let

$$\Gamma_n = \{(x, f(x)) : n - 1 \leq x < n\}.$$

It suffices to prove that $m_*(\Gamma_n) = 0$ for all $n$ since outer measure is countably subadditive. By translation invariance, it suffices to consider

$$\Gamma_1 = \{(x, f(x)) : 0 \leq x < 1\}.$$

Since this set is contained in

$$\Gamma_1' = \{(x, f(x)) : 0 \leq x \leq 1\},$$

it will be enough to show that $\Gamma_1'$ has measure 0. Since $f$ is continuous on $[0, 1]$, it is uniformly continuous. Thus given any $\varepsilon > 0$ there is a $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. Fix $\varepsilon > 0$ and choose $N$ so that $1/N < \delta$. Dividing $[0, 1]$ into intervals of size $1/N$ we can enclose each set

$$\{(x, f(x)) : (j - 1)/N \leq x < j/N\}$$

in a rectangle of width $2/N$ and height $\varepsilon$. The total area of these $N$ rectangles is $2\varepsilon$. It follows that $m_*(\Gamma_1') < 2\varepsilon$ and, since $\varepsilon > 0$ is arbitrary, $m_*(\Gamma_1') = 0$.

4. (**6 points**) Stein and Shakarchi, page 47, Problem 4.

Let $f$ be abounded function on a compact interval $J$, let

$$I(c, r) = (c - r, c + r),$$

and define

$$\begin{aligned}\operatorname{osc}(f,c,r) &= \sup\{|f(x)-f(y)| : x,y \in J \cap I(c,r)\}\\ \operatorname{osc}(f,c) &= \lim_{r\to 0} \operatorname{osc}(f,c,r)\end{aligned}$$

where the second definition makes sense because $\operatorname{osc}(f,c,r)$ from above and below, and $\operatorname{osc}(f,c,r_1) \leq \operatorname{osc}(f,c,r_2)$ if $0 < r_1 < r > 2$. The function $f$ is continuous at $c$ if and only if $\operatorname{osc}(f,c) = 0$.[1] We will prove:

**Theorem** *A bounded function $f$ on a compact interval $J$ is Riemann integrable if and only if its set of discontinuities has measure zero.*

(a) (**2 points**) For any $\varepsilon > 0$ the set of points in $J$ so that $\operatorname{osc}(f,c) \geq \varepsilon$ is compact.

Let $A_\varepsilon = \{c \in J : \operatorname{osc}(f,c) \geq \varepsilon\}$. Since $J$ is bounded it suffices to show that $A_\varepsilon$ is closed, or equivalently to show that $A_\varepsilon^c$ is open. If $c \in A_\varepsilon^c$, then either $c \in J^c$ (which, as the complement of a compact set, is open), or $c \in J$ but $\operatorname{osc}(f,c) < \varepsilon$. In the first case, $c$ is an interior point of $J_\varepsilon^c$, hence an interior point of $A_\varepsilon^c$. In the second case, there is a $\delta > 0$ so that $\operatorname{osc}(f,c,\delta) < \varepsilon$. Since $\operatorname{osc}(f,c,r)$ is monotone nonincreasing in $r$, it follows that for any $c'$ with $|c-c'| < \delta/4$, $\operatorname{osc}(f,c') < \varepsilon$, so again $c$ is an interior point of $A_\varepsilon^c$.

(b) (**2 points)** Suppose that the set of discontinuities $A$ of $f$ has measure zero and that $|f(x)| \leq M$ for all $x \in J$. Consider the sets $A_{1/n}$ where $A_\varepsilon$ was defined above. The set $A_{1/n}$ is compact and has measure zero since $A_{1/n} \subset A$. Fix $n$. Given any $\varepsilon > 0$ (unrelated to $n$) we can cover $A_{1/n}$ by a countable union of intervals with total length $\leq \varepsilon$. By compactness we can extract a finite subcover, say $\{I_n\}_{n=1}^N$, with $\sum_{n=1}^N |I_n| < \varepsilon$, and close the intervals to obtain a set of closed intervals containing $A_{1/n}$ which we may assume are disjoint. Let $B = \cup_{i=1}^N \overline{I_n}$. On $J \setminus B$ we have $\operatorname{osc}(f,c) < 1/n$, so for each $c \in J \setminus B$ there is an $r > 0$ so that $\operatorname{osc}(f,c,r) < 2/n$. Thus we can partition $J \setminus B$ into intervals $[x_{j-1}, x_j]$ with $M_j - m_j < 2/n$, where $M_j$ and $m_j$ are

[1]Students are allowed to assume this, but here's a proof. If $\operatorname{osc}(f,c) = 0$, given any $\varepsilon > 0$ there is a $\delta > 0$ so that for any $x,y \in I(c,\delta)$, $|f(x)-f(c)| \leq \operatorname{osc}(f,c,r) < \varepsilon$. On the other hand, if $f$ is continuous at $c$ then for any $\varepsilon > 0$ there is a $\delta > 0$ so that $|f(x)-f(c)| < \varepsilon$ if $|x-c| < \delta$. Estimating $|f(x)-f(y)| \leq |f(x)-f(c)| + |f(y)-f(c)|$ we see that $\operatorname{osc}(f,c,\delta) < 2\varepsilon$, which shows that $\operatorname{osc}(f,c,r) \to 0$ as $r \to 0$ since $\operatorname{osc}(f,c,r)$ is monotone decreasing.

the maximum and minimum values of $f$ on $[x_{j-1}, x_j]$. Let $P$ be a partition consisting of the $\{\overline{I}_i\}$ and the small intervals in $J \setminus B$. We may estimate

$$\begin{aligned} U(f,P) - L(f,P) &\leq \sum_j (M_j - m_j)(x_{j+1} - x_j) + 2M \sum_{i=1}^{N} |I_i| \\ &\leq (2/n) + 2M\varepsilon \end{aligned}$$

Choosing $n$ large enough we can find a partition $P$ so that $U(f,P) - L(f,P) < 4M\varepsilon$ and, since $\varepsilon > 0$ is arbitrary, we conclude that $f$ is Riemann integrable. This proves the first direction.

(c) (**2 points**) Suppose that $f$ is Riemann integrable. We will use this fact to estimate the measure of the discontinuity set. Since $f$ is integrable, there is a partition $P$ so that $U(f,P) - L(f,P) < \varepsilon/n$. Denote by $I_j$ the $j$th interval in this partition. We may estimate

$$\begin{aligned} \frac{1}{n} m(A_n) &\leq \sum_{j : I_j \cap E \neq \emptyset} \frac{1}{n} |I_j| \\ &\leq \sum_{j : I_j \cap E \neq \emptyset} (M_j - m_j)|I_j| \\ &\leq U(f,P) - L(f,P) \\ &< \varepsilon/n \end{aligned}$$

and conclude that $m(A_{1/n} < \varepsilon$ for any $\varepsilon > 0$. Since $\varepsilon > 0$ is arbitrary, we conclude that $m(A_{1/n}) = 0$ for all $n$. Since $A$, the set of discontinuities of $f$, is given by $A = \cup_{n=1}^{\infty} A_{1/n}$, it follows that $m(A) = 0$.