PROBLEM SET #5

1. (**Not graded**) Stein and Shakarchi, page 89, Exercise 1

Consider the 2^n sets F_j^* where $F_j^* = F_1' \cup F_2' \cup \ldots \cup F_n'$ where each F_k', $1 \leq k \leq n$, is either F_k or F_k^c. We'll leave out $F_1^c \cap \ldots \cap F_n^c$ which contains no points of any F_j, leaving $2^n - 1$ sets F_k'.

First, we claim that the F_j' are disjoint. If $j \neq k$ then there is at least one index i, $1 \leq i \leq n$, so that, writing

$$F_j' = F_1' \cup \ldots \cup F_i' \cup \ldots \cup F_n'$$

and

$$F_k' = F_1'' \cup \ldots \cup F_i'' \cup \ldots \cup F_n'',$$

we have $F_i' = (F_i'')^c$. It follows that no x can belong both to F_j' and to F_k'.

Next, we claim that

$$\cup_{k=1}^n F_k = \cup_{j=1}^{2^n-1} F_j'.$$

If we fix k, we can obtain F_k by taking the union over all F_j' of the form

$$F_j' = F_1' \cup \ldots \cup F_k \cup \ldots \cup F_n'$$

(i.e., only the kth set is fixed). Note that, by this construction,

$$F_j = \cup_{k' : F_k' \subset F_j} F_k'.$$

It follows that $\cup_{k=1}^n F_k \subset \cup_{j=1}^{2^n-1} F_j'$. On the other hand, any $x \in \cup_{k=1}^{2^n-1} F_k'$ must lie in at least one F_k, so $\cup_{k=1}^{2^n-1} F_k' \subset \cup_{k=1}^n F_k$. This shows that $\cup_{k=1}^n F_k = \cup_{j=1}^{2^n-1} F_j'$ as claimed.

Due: February 25, 2019.

2. (**2 points**) Stein and Shakarchi, page 91, Exercise 9 (Tchebychev inequality)

   Suppose that $f \geq 0$ and that f is integrable, and let
$$E_\alpha = \{x : f(x) > \alpha\}.$$
   Then
$$\begin{aligned} \alpha m(E_\alpha) &= \int_{E_\alpha} \alpha && \text{(Integration of simple functions)} \\ &\leq \int_{E_\alpha} f && \text{(Monotonicity)} \\ &\leq \int f && \text{(Monoticity again)} \end{aligned}$$
   Hence
$$m(E_\alpha) \leq \frac{1}{\alpha} \int f$$
   which is Tchebyshev's inequality.

3. (**8 points**) Stein and Shakarchi, page 91, Exercise 10

   (a) (**4 points**)
   Suppose that $f \geq 0$. Let
$$E_{2^k} = \left\{x : f(x) > 2^k\right\}$$
   and
$$F_k = \left\{x : 2^k < f(x) \leq 2^{k+1}\right\}.$$
   We claim that the following three statements are equivalent.

   (1) f is integrable
   (2) $\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$
   (3) $\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty$

   We will show (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3).

   Suppose that f is integrable. We may estimate
$$\begin{aligned} 2^k m(F_k) &= \int_{F_k} 2^k && \text{(integration of simple functions)} \\ &< \int_{F_k} f && \text{(montonicity)} \end{aligned}$$
   The sums $\sum_{k=-N}^{N} \int_{F_k} f$ are monotone nondecreasing in N and bounded above by $\int f$. It follows from the equality above that

$\sum_{k=1}^{\infty} 2^k m(F_k)$ converges, showing that (1) $\Rightarrow$ (2). On the other hand, we may estimate

$$\begin{aligned} \int_{\cup_{k=-N}^{N} F_k} f &\leq \sum_{k=-N}^{N} \int_{F_k} f \\ &\leq \sum_{k=-N}^{n} 2^{k+1} m(F_k) \end{aligned}$$

By Fatou's lemma, $\int f \leq \liminf_{N\to\infty} \int_{\cup_{k=-N}^{N}} f < \infty$, so $\int f$ is finite, showing that (2) $\Rightarrow$ (1).

Next, note that

$$E_{2^k} = \cup_{j=k}^{\infty} F_k$$

so that

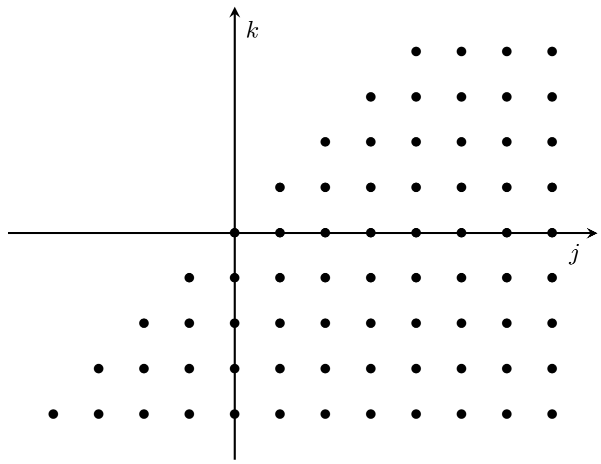
$$m(E_{2^k}) = \sum_{j=k}^{\infty} m(F_j)$$

where the right-hand sum converges by the convergence of $\sum_{k=-\infty}^{\infty} 2^k m(F_k)$ and the comparison test. Now consider the sum

$$\begin{aligned} \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) &= \sum_{k=-\infty}^{\infty} \sum_{j=k}^{\infty} 2^k m(F_j) \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{j} 2^k m(F_j) \\ &= \sum_{j=-\infty}^{\infty} 2^{j+1} m(F_j) \\ &= 2 \sum_{j=-\infty}^{\infty} 2^j m(F_j) \\ &< \infty \end{aligned}$$

(see Figure 1 to understand the second step). This shows that (2) $\Rightarrow$ (3).

FIGURE 1. Ordered pairs occuring in $\sum_{k=-\infty}^{\infty}\sum_{j=k}^{\infty}(\ldots)$



Finally, suppose that $\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty$. We compute

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) = \sum_{k=-\infty}^{\infty} 2^k \left[m(E_{2^k}) - m(E_{2^{k+1}})\right]$$
$$= \sum_{k=-\infty}^{\infty} 2^{k-1} m(E_{2^k})$$

which is clearly convergent, showing that (3) $\Rightarrow$ (2).

(b) (**4 points**)
Now consider the function

$$f(x) = \begin{cases} |x|^{-a}, & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

For this function,

$$E_{2^k} = \{x : |x| < 1,\ |x|^{-a} > 2^k\} = \begin{cases} \{x : |x| < 2^{-k/a}\} & k \ge 1 \\ \{x : |x| < 1\} & k \le 0 \end{cases}$$

so

$$m(E_k) = \begin{cases} c_d 2^{-dk/a}, & k \ge 0 \\ 1 & k < 0 \end{cases}$$

Hence

$$\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) = \sum_{k=-\infty}^{0} 2^k + c_d \sum_{k=1}^{\infty} 2^{(1-d/a)k}$$

which converges so long as $a < d$.

Finally, consider the function

$$g(x) = \begin{cases} |x|^{-b}, & |x| > 1 \\ 0 & \text{otherwise.} \end{cases}$$

We compute

$$E_{2^k} = \{x : |x| \geq 1,\ |x|^{-b} > 2^k\} = \begin{cases} \{x : 1 \leq |x| < 2^{-k/b}\} & k < 0 \\ \varnothing, & k \geq 0 \end{cases}$$

Hence,

$$m(E_{2^k}) = \begin{cases} 0 & k > 0 \\ c_d \left[(2^{-dk/b} - 1\right] & k \leq 0 \end{cases}$$

and

$$\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) = \sum_{k=-\infty}^{-1} c_d 2^{(1-d/b)k}$$

which converges provided $b > d$.