PROBLEM SET #7

1. **(2 points)** Stein and Shakarchi, page 93, problem 18

   Solution. Let $F(x,y) = |f(x) - f(y)|$.Since F is integrable on $[0,1] \times [0,1]$, $F^y(x) = |f(x) - f(y)|$ is integrable on $[0,1]$ for a.e. y. Since f is finite-valued we may pick one such y, say $y = a$, and setting $f(a) = b$ we conclude that $|f(x) - b|$ is integrable, hence that f is integrable.

2. **(2 points)** Stein and Shakarchi, page 93, problem 19. *Hint*: Compute the measure of the set

$$E = \{(x,y) : 0 \le y \le f(x)\}$$

two ways, using Fubini's theorem.

Ex Post Facto Hint:[1] In fact you can take

$$E = \{(x,y) : 0 \le y < f(x)\}.$$

Solution. Following the hint let χ_E be the characteristic function of the set E. Note that

$$E^y = \left\{x \in \mathbb{R}^d : f(x) < y.\right\}.$$

Then

$$\begin{aligned} m(E) &= \int \chi_E(x,y)\,dx\,dy \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} \chi_E(x,y)\,dx \right) dy \\ &= \int_{\mathbb{R}} m(E^y)\,dy \\ &= \int_0^\infty m(E^y)\,dy \end{aligned}$$

Due: March 25, 2019.

[1] I am indebted to one of you for copious correspondence. You know who you are!

where in the last step we used the fact that $E^y = \varnothing$ for $y < 0$. On the other hand

$$m(F) = \int_{\mathbb{R}^d} \left(\int_R \chi_E(x,y)\, dy \right) dx$$
$$= \int_{\mathbb{R}^d} f(x)\, dx$$

as was to be proved.

3. (**6 points**) Stein and Shakarchi, page 94, problem 21 (a) - (c). For (a), recall Corollary 3.7 and Proposition 3.9.

(a) (**2 points**) Prove that $f(x-y)g(y)$ is measurable on $\mathbb{R}^{2d}$.

Solution Both f and g are finite a.e. since they are integrable functions. From Property 5 of chapter 1 (see the remarks before Property 6 to address the problem that f and g are finite-valued almost everywhere rather than everywhere) it suffices to show that $F(x,y) = f(x-y)$ and $G(x,y) = g(y)$ are measurable. This is an immediate consequence of Proposition 3.9 (for F) and Corollary 3.7 (for G).

Remark: There is one small subtlety here. In order to apply Property 5 and the remarks before property 6, we need to show that F and G are finite a.e. when viewed as functions on $\mathbb{R}^d \times \mathbb{R}^d$. Thus if Z_f and Z_g are the subsets of measure zero in $\mathbb{R}^d$ on which f and g are not finite-valued, we need to know that the sets

$$\widetilde{Z_f} = \{(x,y) : x - y \in Z_f\}$$
$$Z_g \times \mathbb{R}^d = \left\{(x,y) : x \in \mathbb{R}^d, y \in Z_g\right\}$$

are also sets of measure zero. One can show that $\widetilde{Z_f}$ has measure zero by considering the sets $\widetilde{Z_f} \cap B_k$ where $B_k = \{y \in \mathbb{R}^d : |y| < k\}$ exactly as in the proof of Proposition 3.9. Using the fact that $Z_g \times B_k \nearrow Z_g \times \mathbb{R}^d$ and $m(Z_g \times B_k) = m(Z_g)m(B_k) = 0$ we can conclude that $Z_g \times \mathbb{R}^d$ also has measure zero.

(b) (**2 points)** Show that if f and g are integrable on $\mathbb{R}^d$, then $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2d}$.

Solution It is enough to show that $\int_{\mathbb{R}^{2d}} |f(x-y)||g(y)|\,dx\,dy < \infty$. By Tonelli's theorem we compute

$$\begin{aligned}\int_{\mathbb{R}^{2d}} &|f(x-y)||g(y)|\ dx\,dy \\ &= \int_{\mathbb{R}^d} |g(y)| \left(\int_{\mathbb{R}^d} |f(x-y)|\,dx\right)\ dy \\ &= \left(\int_{\mathbb{R}^d} |g(y)|\,dy\right)\left(\int_{\mathbb{R}^d} |f(x)|\,dx\right) \\ &< \infty.\end{aligned}$$

In the second step, we used the fact that

$$\int |f(x-y)|\,dx = \int |f(x)|\,dx$$

by translation invariance of Lebesgue measure.[2]

(c) (**2 points**) Show that the convolution of integrable functions f and g, defined by

$$(f*g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy$$

is well-defined for a.e. x

Solution Let $H(x,y) = f(x-y)g(y)$. In part (a) we showed that H is measurable and in part (b) we showed that H is integrable. It follows from Fubini's Theorem that the slice $H_x(y)$ is measurable and integrable on $\mathbb{R}^d$ as a function of y for a.e. x. Since $(f*g)(x) = \int H_x(y)\,dy$, it follows that $(f*g)(x)$ is defined for a.e. x.

[2]This follows from the fact that, if E is a measurable set of finite measure and $E+y$ denotes the set $\{x+y : x \in E\}$, then $m_*(E+y) = m_*(E)$.