MATH 676
PROPERTIES OF MEASURABLE FUNCTIONS

1. Measurable Functions

Definition 1. We say that $f : E \to [-\infty, \infty]$ is *finite-valued* if $-\infty < f(x) < \infty$ for all $x \in E$.

Definition 2. Let $f : E \subset \mathbb{R}^d \to [-\infty, \infty]$ where the domain, E, is a measurable set in $\mathbb{R}^d$. We say that f is *measurable* if for every $a \in \mathbb{R}$, the set

$$f^{-1}([-\infty, a)) = \{x \in E : f(x) < a\}$$

is mesurable.

Definition 3. Suppose f and g are defined on a subset E of $\mathbb{R}^d$. Then $f = g$ a.e. ("almost everywhere") if the set

$$\{x \in E : f(x) \neq g(x)\}$$

has measure zero.

2. Properties of Measurable Functions

These properties follow from the definitions and the fact that the Lebesgue-measurable sets form a σ-algebra containing the open sets.

Property 1. The following are equivalent for a finite-valued function f.

(i) f is measurable
(ii) $f^{-1}(\mathcal{O})$ is measurable for every open set $\mathcal{O}$.
(iii) $f^{-1}(F)$ is measurable for every closed set F.

Property 2. If f is continuous on $\mathbb{R}^d$, then f is measurable. If f is measurable and finite-valued, and Φ is continuous, then $\Phi \circ f$ is measruable.

Property 3. Suppose $\{f_n\}$ is a sequence of measurable functions. Then

$$\sup_n f_n(x), \quad \inf_n f_n(x), \quad \limsup_{n\to\infty} f_n(x), \quad \liminf_{n\to\infty} f_n(x)$$

are measurable

Property 4. If $\{f_n\}$ is a sequence of measurable functions, and

$$\lim_{n\to\infty} f_n(x) = f(x),$$

then f is measurable.

Date: February 4, 2019.

Property 5. If f and g are measurable, then

(i) The integer powers f^k, $k \geq 1$, are measurable.
(ii) $f + g$ and fg are measurable, provided that f and g are both finite-valued.

Property 6. Suppose f is measurable and $f = g$ a.e. Then g is measurable.