MATH 676
OVERVIEW OF LEBESGUE INTEGRATION

1. Step One: Simple Functions

A simple function in *canonical form* is a sum

$$\varphi(x) = \sum_{k=1}^{N} a_k \chi_{E_k}$$

where a_k are real numbers and the sets $\{E_k\}$ are measurable and *disjoint*. For such a function we define

$$\int \varphi \, dm = \sum_{k=1}^{N} a_k m(E_k).$$

If E is a measurable set we define

$$\int_E \varphi \, dm \int \chi_E \varphi \, dm$$

(why does this definition make sense)?

The Lebesgue integral on simple functions has the following properties. Dnoete by φ and ψ simple functions and by a, b real numbers.

(i) (Independence of Representation) If $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ is any representation of φ, then

$$\int \varphi = \sum_{k=1}^{N} a_k m(E_k).$$

(ii) (Linearity)

$$\int (a\varphi + b\psi) = a \int \varphi + b \int \psi.$$

(iii) (Additivity) If E and F are disjoint measurable sets of $\mathbb{R}^d$ with finite measure, then

$$\int_{E \cup F} \varphi = \int_E \varphi + \int_F \varphi$$

(iv) (Monotonicity) If $\varphi leq \psi$ then

$$\int \varphi \leq \int \psi$$

(v) (Triangle Inequality) If φ is a simple function, so is $|\varphi|$, and

$$\left| \int \varphi \right| \leq \int |\varphi|.$$

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2. Step Two: Bounded Functions Supported on a Set of Finite Measure

The support of a measurable function f is the set

$$\operatorname{supp}(f) = \{x : f(x) \neq 0\}.$$

We extend the Lebesgue integral to bounded functions supported on a set of finite measure.

Lemma 1. *Suppose f is a bounded measurable function with support on a set E of finite measure If $\{\varphi_n\}$ is any sequence of simple functions converging off for a.e. x:*

(i) The limit $\lim_n \int \varphi_n$ exists
(ii) If $f = 0$ a.e., then $\lim_n \int \varphi_n = 0$.

The proof uses Egorov's theorem.

Definition 1. If f is a bounded measurable function whose support is a set of measure, we define

$$\int f(x)\,dx = \lim_{n\to\infty} \int \varphi_n(x)\,dx$$

where $\{\varphi_n\}$ is any sequence of simple functions satisfying (i) $|\varphi_n(x)| \leq M$ for some $M > 0$ independent of n, (ii) φ_n has support contained in the support of f, and (iii) $\varphi_n(x) \to f(x)$ for a.e. x.

One checks that properties (ii) – (v) of the Lebesgue integral on simple functions carry over.

3. Step Three: Nonnegative Functions

Definition 2. If f is a nonnegative measurable function taking values in $[0, \infty]$ we define

$$\int f(x)\,dx = \sup_g \int g(x)\,dx$$

where the supremem is taken over all bounded measurable functions g with support on a set of bounded measure having the property that $0 \leq g \leq f$.

Definition 3. We say that f is *integrable* if $\int f < \infty$.

Theorem 1. *This extension of the Lebesgue integral has the following properties:*

(i) If $f \geq 0, g \geq 0$ and a, b are positive real numbers,

$$\int (af + bg) = a \int f + b \int g$$

(ii) If E and F are disjoint measurable sets and $f geq 0$, then

$$\int_{E\cup F} f = \int_E f + \int_F f.$$

(iii) If $0 \leq f \leq g$ then

$$\int f \leq \int g.$$

(iv) If g is integrable and $f \leq g$, then f is integrable.
(v) If f is integrable, then $f(x) < \infty$ for a.e. x.
(vi) If $\int f = 0$, then $f(x) = 0$ for a.e. x.

A very important result in the theory is

Lemma 2 (Fatou's Lemma)**.** *Suppose* $\{f_n\}$ *is a sequence of nonnegative measurable functions. If* $\lim_{n\to\infty} f_n(x) = f(x)$ *for a.e.* x*, then*

$$\int f \leq \liminf_{n\to\infty} \int f_n.$$

An important corollary is:

Theorem 2 (Monotone Convergence)**.** *Suppose that* $\{f_n\}$ *is a sequence of nonnegative measurable functions with* $f_n \nearrow f$*. Then*

$$\lim_{n\to\infty} \int f_n = \int f.$$

4. Step Four: General Case

Definition 4. A measurable function f is *Lebesgue integrable* if $|f|$ is integrable in the sense of Definition 3. We then define

$$\int f = \int f_+ - \int f_-$$

Theorem 3. *The Lebesgue integral is linear, additive, monotonic, and satisfies the triangle inequality. Moreover, if* f *is integrable:*

(i) For every $\varepsilon > 0$*, there is a set* B *of finite measure so that* $\int_{B^c} |f| < \varepsilon$
(ii) There is a $\delta > 0$ *so that*

$$\int_E f < \varepsilon \text{ whenever } m(E) < \delta$$

Theorem 4 (Dominated Convergence)**.** *Suppose that* $\{f_n\}$ *is a sequence of measurable functions so that* $f_n(x) \to f(x)$ *for almost every* x *and* $|f_n(x)| \leq g(x)$ *for a fixed integrable function* g*. Then*

$$\lim_{n\to\infty} f_n = \int f.$$