OVERVIEW OF MEASURE THEORY AND INTEGRATION

Outer Measure and Lebesgue Measure

If $E \subset \mathbb{R}^d$, the *outer measure* of E is the nonnegative extended real number

$$m_*(E) = \inf \left\{ \sum_{i=1}^{\infty} |Q_i| : E \subset \bigcup_{i=1}^{\infty} E_i \right\}$$

We will show that $m_*(\,\cdot\,)$ is a monotone, countably subadditive set function.

We will call a subset E of $\mathbb{R}^d$ *Lebesgue measurable* (or simply *measurable*) if for each $\varepsilon > 0$ there is an open set $\mathcal{O} \supset E$ with

$$m_*(\mathcal{O} - E) \le \varepsilon.$$

This is a "regularity condition" on E which says that it can be approximated 'from the outside' by open sets.

If E is measurable, the Lebesgue measure of E, denoted by $m(E)$, is given by $m(E) = m_*(E)$. That is, m is the restriction of the set function m_* to the Lebesgue measurable sets.

Denote by $\mathfrak{M}$ the collection of Lebesgue measurable sets. Obvously, $\mathfrak{M}$ contains the open sets. We will show that $\mathfrak{M}$ is closed under countable unions, countable intersections, and complements (such a collection of sets is called a σ-algebra). and will show that m is well-behaved under these operations.

In what follows, we'll say that a property holds *almost everywhere* (abbreviated a.e.), or for *almost every* x (abbrevitated a.e. x), if the set on which the property does *not* hold has Lebesgue measure zero.

Measurable Functions

A function $f : D \subset \mathbb{R}^d \to \mathbb{R}$ is *measurable* if, for every $a \in \mathbb{R}$, the set $E_a = \{x \in D : f(x) < a\}$ is a measurable set. Notice that, since $\mathfrak{M}$ contains open sets, any continuous function is necessarily measurable. Also notice that the characteristic function of a measurable set is measurable.

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By the time we finish with measurable functions, we will understand (and formulate precisely) Littlewood's[1] three principles:

(1) Every measurable set is 'nearly' a finite union of intervals
(2) Every measurable function is 'nearly' a continuous function (Lusin's Theorem)
(3) Every convergent sequence of measurable functions is 'nearly' uniformly convergent (Ěgorov's Theorem)

THE LEBESGUE INTEGRAL

We will develop a generalization of the Riemann integral for measurable functions f, the *Lebesgue integral* of f, which agress with the Riemann integral if f is a Riemann-integrable function. We'll begin by integrating *simple functions*, i.e., functions of the form

$$f(x) = \sum_{i=1}^{n} c_i \chi_{E_i}$$

where the c_i are real numbers and χ_E is the characteristic function of the measurable set E, that is

$$\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E. \end{cases}$$

We'll *define*

$$\int f = \sum_{i=1}^{n} c_i m(E_i).$$

From this starting point, we will work up successfully through:

(1) Simple functions
(2) Bounded measurable functions with support on a set of finite measure
(3) Nonnegative functions
(4) 'Integrable' functions

A measurable function f is called *integrable* if $\int |f|$ is finite. The set of all integrable functions is denoted $\mathcal{L}^1(\mathbb{R}^d)$. It will be useful to quotient out by the following equivalence relation: $f \sim g$ if f and g are measurable and $f(x) - g(x) = 0$ for a.e. x. The quotient of $\mathcal{L}^1(\mathbb{R}^d)$ by this equivalence relation is denoted $L^1(\mathbb{R}^d)$. We will show that $L^1(\mathbb{R}^d)$ is a complete metric space with metric $d(f,g) = \int |f - g|$.

[1]J. E. Littlewood (1885-1977) was an English mathematician who worked extensively with G. H. Hardy and was thesis advisor for Srinivasa Ramanujan.

We will prove three key convergence theorems for the Lebesgue integral.

(1) **Fatou's Lemma:** If $\{f_n\}$ is a sequence of nonnegative measurable functions and $f_n(x) \to f(x)$ for almost every x, then $\int f \leq \liminf_{n\to\infty} \int f_n$.
(2) **Monotone Convergence Theorem**: If $\{f_n\}$ is a sequence of nonnegative measurable functions with $f_n(x) \leq f_{n+1}(x)$ for a.e. x and $f_n \to f$ a.e., then
$$\int f = \lim_{n\to\infty} \int f_n.$$
(3) **Dominated Convergence Theorem**: If $\{f_n\}$ is a sequence of measurable functions with $f_n \to f$ a.e and $|f_n(x)| \leq g(x)$ for a fixed, nonnegative, integrable function g, then $\lim_{n\to\infty} \int f_n = \int f$.