MATH 676
OBSERVATIONS AND PROPERTIES

1. Observations on Outer Measure

Recall that the *outer measure* of a subset E of $\mathbb{R}^d$ is defined as follows:

$$\boxed{m_*(E) = \inf\left\{\sum_{j=1}^{\infty} |Q_j| : E \subset \cup_{j=1}^{\infty} Q_j\right\}}$$

The goal of this development is to deduce properties of outer measure that will help us construct Lebesgue measure.

Observation 1 (Monotonicity)**.** If $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$

Observation 2 (Countable subadditivity)**.** If $E = \cup_{j=1}^{\infty} E_j$, then

$$m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$$

Observation 3 (Regularity)**.** If $E \subset \mathbb{R}^d$, then

$$m_*(E) = \inf\left\{m_*(\mathcal{O}) : \mathcal{O} \supset E, \mathcal{O} \text{ is open }\right\}.$$

Observation 4. If $E = E_1 \cup E_2$ and $d(E_1, E_2) > 0$, then

$$m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2)$$

Observation 5. If E is a countable union of almost disjoint cubes Q_j, then

$$m_*(E) = \sum_{j=1}^{\infty} |Q_j|.$$

Date: January 16, 2019.

2. Properties of Lebesgue Measure

Recall that

> A subset E of $\mathbb{R}^d$ is *Lebesgue measurable* (or *measurable*) if for every $\varepsilon > 0$ there is an open set $\mathcal{O}$ so that
> $$m_*(\mathcal{O} - E) \leq \varepsilon.$$

The goal of this development is to show that the Lebesgue measurable sets form a σ-algebra that contains the open sets. Recall that a σ-algebra of sets is a collection of sets that is closed under countable unions, countable intersections, and complements.

Property 1. All open sets are measurable.

Property 2. If $m_*(E) = 0$ then E is measurable. In particular, if F is a subset of a set of exterior measure 0, then F is measurable.

Property 3. A countable union of measurable sets is measurable.

Property 4. Closed sets are measurable.

Property 5. The complement of a measurable set is measurable.

Property 6. A countable intersection of measurable sets is measurable.