MATH 676
FURTHER PROPERTIES OF LEBESGUE MEASURE

Recall that the outer measure of a set $E \subset \mathbb{R}^d$ is given by

$$m_*(E) = \inf \left\{ \sum_{j=1}^{\infty} |Q_i| : E \subset \cup_{j-1}^{\infty} Q_j \right\}$$

where the infimum is over covers of E by closed cubes.

Recall that a subset E of $\mathbb{R}^d$ is Lebesgue measurable if for any $\varepsilon > 0$, there is an open set $\mathcal{O} \supset E$ with

$$m_*(\mathcal{O} - E) \leq \varepsilon.$$

We have shown that the measurable sets contain open sets and are closed under countable unions, countable intersections, and complements. It remains to prove countable additivity and several *monotonicity* and *regularity* properties of Lebesgue measure.

1. Countable Additivity

Theorem 1. *If $\{E_i\}_{i=1}^{\infty}$ are disjoint measurable sets, and $E = \cup_{i=1}^{\infty} E_i$, then*

$$m(E) = \sum_{j=1}^{\infty} m(E_j).$$

1.1. **Monotonicity.** If $\{E_k\}_{k=1}^{\infty}$ is a sequence of measurable sets with $E_k \subset E_{k+1}$ and $E = \cup_{k=1}^{\infty} E_k$, we say that $E_k \nearrow E$.

If $\{E_k\}_{k=1}^{\infty}$ is a sequence of measurable sets with $E_k \supset E_{k+1}$ and $E = \cup_{k=1}^{\infty} E_k$, we say that $E_k \searrow E$.

Theorem 2. *Suppose $\{E_k\}$ is a sequence of measurable sets.*

(i) f $E_k \nearrow E$, then $m(E_k) \to m(E)$.
(ii) If $E_k \searrow E$, then $m(E_k) \to m(E)$.

1.2. **Regularity.** If E and F are subsets of $\mathbb{R}^d$, the *symmetric difference* of E and F, denoted $E \Delta F$, is given by

$$E \Delta F = (E - F) \cup (F - E).$$

Theorem 3. *Suppose that $E \subset \mathbb{R}^d$ is measurable. Then for every $\varepsilon > 0$:*

(i) There is an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m(\mathcal{O} - E) \leq \varepsilon$.
(ii) There is a closed set F with $F \subset E$ and $m(E - F) \leq \varepsilon$.
(iii) If $m(E)$ is finite, there is a compact set K with $K \subset E$ and $m(E - K) < \varepsilon$.
(iv) If $m(E)$ is finite, there is a finite union $F = \cup_{i=1}^{N} Q_i$ of closed cubes such that

$$m(E \Delta F) \leq \varepsilon.$$

Date: January 23, 2019.

An F_σ set is a countable union of closed sets, and a G_δ set is a countable intersection of open sets. Using (i) and (ii) of the previous theorem, we can prove:

Theorem 4. *A subset E of $\mathbb{R}^d$ is measurable:*

(i) iff E differs from a G_δ set by a set of measure zero
(ii) iff E differs from a an F_σ set by a set of measure zero.