

Imaging with electricity: the mathematics of electrical
impedance tomography

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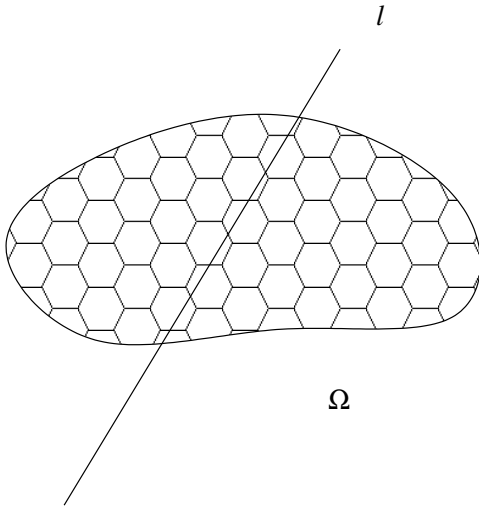
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Outline.

1. The Radon transform and a little bit about CAT scanning.
2. Electrical impedance tomography.

Part I: The Radon transform

Let $\Omega \subset \mathbf{R}^n$ and let $\rho : \Omega \rightarrow \mathbf{R}$ be a function describing the attenuation of x-rays in Ω . If we shine an x-ray beam along a line, ℓ , passing through Ω and measure the decrease in intensity of the beam as it passes through Ω , this amounts to measuring the integral of ρ along ℓ :



$$R(\rho)(\ell) = \int_{\ell} \rho ds.$$

Can we recover ρ from $R(\rho)$?

The function $R(\rho)(\ell)$ is called the **Radon transform** of f . In order to be able to recover ρ from its Radon transform $R(\rho)$, we need the map

$$\rho \rightarrow R(\rho)$$

to be one-to-one. Even better, we would like a formula that shows us how to compute the attenuation function from its Radon transform.

This is the mathematical question that we need to be able to answer in order to construct images in a medical CAT (computer assisted tomography) scanner.

Yes, CAT scanners work.

There is an explicit formula for recovering ρ from $R(\rho)$. (But this is another talk.) This was discovered by Johann Radon (1887–1956).

A. Cormack and G. Hounsfield shared the 1979 Nobel prize in physiology and medicine for developing CAT scanning technology.

From the press release announcing Cormack's Nobel prize:

He was the first, from a theoretical point of view, to analyze the conditions for demonstrating a correct radiographic cross-section in a biological system. He published his analysis of the problem in two scientific publications in 1963 and 1964. He understood that the problem was basically a mathematical one. It was a matter of finding a reasonable two-dimensional function that described how X-rays attenuate in each individual part within a slice when one knows the mean values of the rays' absorption, the so-called line integrals, along a number of straight lines within this slice. . . . He was not aware then that the key mathematical problems had been considered earlier in an altogether different connection and deduced his own method of calculation.

Cormack shared the Nobel prize with Hounsfield and the press release continues:

Hounsfield was obviously unaware of Cormack's contributions and developed his own method for reconstruction of the image.

Three dimensions.

Notice that in two dimensions, the family of all lines in the plane is two-dimensional. A generic line is described by its slope and y -intercept.

Three dimensions is different. A generic line in three space is described by (say) the point where it intercepts the plane $\{z = 0\}$ and a unit direction vector. A point on the plane is described by two coordinates and the direction vector corresponds to a point on the two-dimensional unit sphere, so the family of all lines is four dimensional.

The problem of finding ρ from the Radon transform is overdetermined in three dimensions.

Three dimensions continued.

Practically, this means that measuring the full Radon transform in three dimensions would involve exposing the patient to unnecessary radiation.

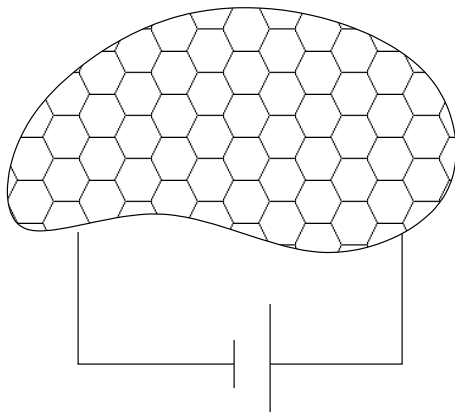
Solution 1. Measure a series of two-dimensional slices.

Solution 2. Helical CT scanning.

Part II: Electrical impedance tomography (EIT)

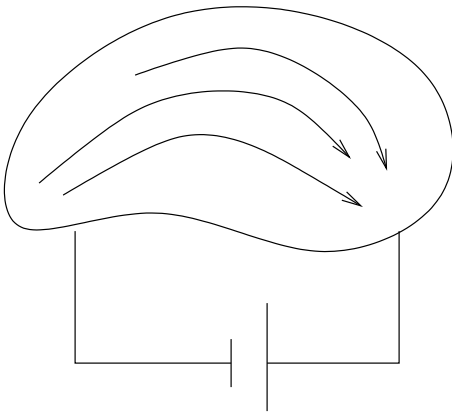
In electrical impedance tomography, one is trying to find a function $\gamma(x)$ which represents the conductivity at a point x in a region Ω in two or three dimensions.

One makes measurements by measuring the voltage potential needed to induce a current to flow between two points on the boundary.



Why EIT is difficult.

With a CAT scan, each measurement contains information about ρ on a line. In contrast, each measurement in EIT contains information about γ throughout the region.



The mathematical formulation.

The voltage potential, u , is a real-valued function in Ω .

The current induced by this potential is $\gamma \nabla u$.

Conservation of charge implies that

$$\operatorname{div} \gamma \nabla u = \sum_{i=1}^n \frac{\partial}{\partial x_i} \gamma(x) \frac{\partial}{\partial x_i} u(x) = 0.$$

The current flow through the boundary is $\gamma \nu \cdot \nabla u$ where ν is the outer unit normal to the boundary of Ω .

We measure the map that takes a current pattern at the boundary to a voltage potential:

$$g = \gamma \nabla u \cdot \nu \rightarrow C(g) = u \text{ (restricted to the boundary)}$$

and ask if we can recover γ from this map.

The main question of electrical imaging.

Can we recover the conductivity γ , from the map C which takes current patterns to voltage patterns?

The map C is often called the Neumann to Dirichlet map.

The answer to this question is yes, for conductivities which are reasonably smooth.

A simple example.

It is easy to see that if an object is made of layers of conducting and non-conducting material, then its interior conductivity can be described from boundary measurements.

Plastic
Metal
Plastic
Metal

An idea of the proof in three dimensions.

From knowledge of the map C , we can measure the following expressions involving solutions u_1, u_2 of the equation $\operatorname{div} \gamma \nabla u = 0$:

$$\begin{aligned} \int_{\partial\Omega} u_1 \gamma \frac{\partial u_2}{\partial \nu} d\sigma &= \int_{\Omega} \gamma \nabla u_1 \cdot \nabla u_2 + u_1 \operatorname{div} \gamma \nabla u_2 dx \\ &= \int_{\Omega} \gamma \nabla u_1 \cdot \nabla u_2 dx. \end{aligned}$$

The first equality above is the Gauss divergence theorem and the second equality holds if u_2 is a solution of $\operatorname{div} \gamma \nabla u_2 = 0$.

Exponential solutions when γ is constant

Suppose that γ is a constant. Consider $u(x) = e^{x \cdot \zeta}$ where ζ is a vector with complex entries. We have

$$\begin{aligned}\operatorname{div} \gamma \nabla u &= \gamma \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} e^{x \cdot \zeta} \\ &= \gamma \zeta \cdot \zeta e^{x \cdot \zeta} \\ &= \gamma (|\operatorname{Re} \zeta|^2 - |\operatorname{Im} \zeta|^2 + 2i \operatorname{Re} \zeta \cdot \operatorname{Im} \zeta) e^{x \cdot \zeta}.\end{aligned}$$

This will be zero if and only if

$$|\operatorname{Re} \zeta| = |\operatorname{Im} \zeta|, \quad \operatorname{Re} \zeta \cdot \operatorname{Im} \zeta = 0.$$

A detour: the Fourier transform

If f is an integrable function on \mathbf{R}^n , then the Fourier transform of f is defined by:

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} e^{-ix \cdot \xi} f(x) dx.$$

Theorem. *The map $f \rightarrow \hat{f}$ is one-to-one.*

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \hat{f}(\xi) e^{ix \cdot \xi} d\xi$$

Thus, if we can determine \hat{f} , then we know f .

Recovering $\hat{\gamma}$ in three dimensions, an idea of Calderón

Pick $\xi \in \mathbf{R}^n$, and $R > 0$. Choose n_1 and n_2 unit vectors so that ξ , n_1 and n_2 are orthogonal to each other. (Now you should see why **three** dimensions important.)

Set

$$\begin{aligned}\zeta_1 &= -i(\xi + \sqrt{R^2 - |\xi|^2}n_1) + Rn_2 \\ \zeta_2 &= -i(\xi - \sqrt{R^2 - |\xi|^2}n_1) - Rn_2\end{aligned}$$

For future reference, we note that

$$\zeta_1 + \zeta_2 = -2i\xi \qquad \zeta_1 \cdot \zeta_2 = -|\xi|^2.$$

Calderón, continued.

We let u_j be a solution to the equation $\operatorname{div}_\gamma \nabla u = 0$ which is close to the solution of the constant coefficient equation, $e^{x \cdot \zeta_j}$ and then we have

$$\int_{\partial\Omega} u_1 \gamma \frac{\partial u_2}{\partial \nu} d\sigma = \int_{\Omega} \gamma(x) \nabla u_1(x) \cdot \nabla u_2(x) dx$$

A miracle!

$$= -|\xi|^2 \int_{\Omega} \gamma(x) e^{-2ix \cdot \xi} dx$$

$$= -|\xi|^2 \hat{\gamma}(2\xi)$$

The miracle is accomplished by letting R go to infinity. This idea was proposed by Calderón in 1980 and the “miracle” was made rigorous in 1987 by Sylvester and Uhlmann and Novikov in independent work.

Some current research

Some questions for further research.

- What are optimal smoothness conditions that permit one to recover the conductivity from the Neumann to Dirichlet map.
- Design numerical algorithms to compute the conductivity from the Neumann to Dirichlet map.
- Find boundaries between regions of differing conductivities without finding the values of γ .
- Use the internal electrical activity in the body for imaging.

- Discrete models involving currents in graphs have been studied.
- A similar imaging question can be posed in the context of elasticity. A satisfactory answer is available in three dimensions, but not in two dimensions.

Experiments

A group at Rensselaer Polytechnic Institute has built a machine and used it to construct two-dimensional images of the human body.

Applications?

This method is cheap and safe but does not work very well.
Possible applications include:

- Finding cracks in metal parts.
- Monitoring hospital patients for fluid in lungs.

Some references.

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M. Cheney, D. Isaacson and J. C. Newell, SIAM Rev. **41** (1999), no. 1, 85–101 (electronic); MR 99k:78017

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