The mixed problem in two-dimensional Lipschitz domains.

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The mixed problem.

We consider domains $\Omega = \{(x_1, x_2) : x_2 > \phi(x_1)\}$ where ϕ satisfies $\|\phi'\|_\infty < \infty$ and, for convenience, $\phi(0) = 0$. We write $\partial\Omega = D \cup N$ with $D = \partial\Omega \cap \{x_1 > 0\}$ and $N = \partial\Omega \cap \{x_1 \leq 0\}$. We consider the following classical boundary value problem,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f_D & \text{on } D \\ \frac{\partial u}{\partial \nu} = f_N & \text{on } N \\ (\nabla u)^* \in L^p(w\, d\sigma) & \end{cases}$$

We will generally require data f_N from $L^p(N, w\, d\sigma)$ and ask that df_D/ds lie in $L^p(D, w\, d\sigma)$. The weights w will be of the form $|x|^\epsilon$.

If $p = 1$, we must replace the L^p-spaces by Hardy spaces.

The non-tangential maximal function

Our estimate for ∇u uses the non-tangential maximal function, $(\nabla u)^*$.

For $x \in \Omega$, we define $\Gamma(x) = \{y : A|y_1 - x_1| < (y_2 - x_2)\}$ for some A, $\|\phi'\|_\infty < A < \infty$. Then for a function u on Ω, we define the non-tangential maximal function by

$$u^*(x) = \sup_{y \in \Gamma(x)} |u(y)|.$$

This function allows us to use the dominated convergence theorem to show that the values of u on surfaces parallel to the boundary converge as we approach the boundary.

An example

If we let $u(r, \theta) = r^{1/2}\cos(\theta/2)$, then

$$\begin{cases} \Delta u = 0 & \text{in } \{(x_1, x_2) : x_2 > 0\} \\ u(x_1, 0) = 0, & x_1 < 0 \\ \frac{\partial u}{\partial \nu}(x_1, 0) = 0, & x_1 > 0 \end{cases}$$

but we do not have $\nabla u(x_1, 0) \in L^2_{loc}(\mathbf{R})$.

Thus, we cannot solve the mixed problem in L^2 with respect to arc-length for smooth domains. (We focus on the local behavior of u and ignore the behavior at infinity.)

Some results

1. Azzam and Kreyszig (1982) found solutions in $C^{2+\alpha}(\bar{\Omega})$ provided Ω is piecewise smooth and N and D meet at sufficiently small angles.

2. Savaré (1997) shows that in a smooth domain, the solution lies in the Besov space $B_{\infty}^{3/2,2}(\Omega)$. (The estimate $\nabla u \in L^2(\partial\Omega)$ is equivalent to $u \in B_2^{3/2,2}$.)

3. RB (1994) studied Lipschitz domains where N and D meet at an angle strictly less than π and established existence of solutions for $L^2(\partial\Omega)$ with respect to surface measure.

   J. Sykes (1999) extended this to L^p for $1 < p < 2$.

The main theorem

Theorem 1 *Let* $\|\phi'\|_\infty < 1$*. There exists a value* $p_0 = p(\|\phi'\|_\infty) > 1$ *so that if* $1 < p < p_0$*, then the mixed problem with data* $f_N \in L^p(N)$ *and* $df_D/ds \in L^p(D)$ *has a unique solution which satisfies*

$$(\nabla u)^* \in L^p(\partial\Omega, d\sigma).$$

Weighted L^2 estimates from the Rellich-(Payne-Weinberger-Pohozaev-....) identity

As we observed above, the mixed problem is not solvable in L^2 for a half-space. However, we can study the problem in weighted L^2 spaces where the weight is of the form $|x|^\epsilon$. The basic estimate will be obtained from the following extension of the Rellich identity which we learned from Luis Escauriaza.

Lemma 1 (Rellich Identity) *Let $\epsilon > -1$ and $\alpha(z) = az^\epsilon \equiv$ (Re(az^ϵ),Im(az^ϵ)) for some $a \in \mathbf{C}$. Here, $z = x_1 + ix_2 \equiv (x_1, x_2)$. If u is harmonic in Ω and $(\nabla u)^* \in L^2(\sigma_\epsilon)$, then we have*

$$\int_{\partial\Omega} |\nabla u|^2 \alpha \cdot \nu - 2\alpha \cdot \nabla u \frac{\partial u}{\partial \nu}\, d\sigma = 0.$$

To establish this lemma, we observe that div$(|\nabla u|^2\alpha - 2\nabla u \cdot \alpha \nabla u) = 0$ and use the divergence theorem.

Use of the Rellich identity

To make use of this Lemma, we will need to find a vector field $\alpha = (\mathrm{Re}\, az^{\epsilon}, \mathrm{Im}\, az^{\epsilon})$ so that

$$\begin{aligned} \alpha \cdot \nu &\geq c|z|^{\epsilon}, & \text{on } N \\ \alpha \cdot \nu &\leq -c|z|^{\epsilon}, & \text{on } D. \end{aligned}$$

Then if we rearrange the terms, we will obtain

$$\int_D \left|\frac{du}{ds}\right|^2 d\sigma_\epsilon + \int_N \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma_\epsilon \approx \int_N \left|\frac{du}{ds}\right|^2 d\sigma_\epsilon + \int_D \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma_\epsilon$$

This is the main estimate we need for existence.

Trigonometry

Lemma 2 *Let* $\beta = \operatorname{arctan} \|\phi'\|_\infty > 0$. *Assume* $\beta < \pi/4$. *Then, for* $2\beta/(\pi - 2\beta) < \epsilon < 1$ *there exist* $\beta_0 = \beta_0(\epsilon, M)$, $\beta < \beta_0 < (\pi - 2\beta)/2$, *and a complex number* $a = e^{i\lambda}$ *such that the vector field* $\alpha(x) = (Re(az^\epsilon), Im(az^\epsilon))$ *satisfies*

$$-|x|^\epsilon \leq \alpha(x) \cdot \nu(x) < -|x|^\epsilon \sin(\beta_0 - \beta)\,, \qquad x \in N\,; \qquad (1)$$

$$|x|^\epsilon \geq \alpha(x) \cdot \nu(x) > |x|^\epsilon \sin(\beta_0 - \beta)\,, \qquad x \in D. \qquad (2)$$

The restriction $M < 1$

We continue to use $\alpha = (\text{Re}\, az^\epsilon, \text{Im}\, az^\epsilon)$.

Suppose Ω is a graph domain with Lipschitz constant of 1.

If $\alpha \cdot \nu > 0$ on N, we must have $\arg \alpha(x) \in (-3\pi/4, -\pi/4)$ for $\arg(x) \in (-\pi/4, \pi/4)$. Thus $\epsilon < 1$.

If $\alpha \cdot \nu < 0$ on D, we must have $\arg \alpha(x) > \pi/4$ when $\arg x = 3\pi/4$. Thus $\epsilon > 1$.

Hardy spaces

We recall the atomic definition of weighted Hardy spaces, $H^1(\sigma_{\epsilon'})$.

We say a is an atom for $H^1(\sigma_{\epsilon'})$ if (i) a is supported in an interval on $\partial\Omega$, I, (ii) $\int_I a\, d\sigma = 0$ and (iii) $\|a\|_\infty \le \sigma_{\epsilon'}(I)^{-1}$.

An element of the Hardy space is given by $\sum \lambda_j a_j$ where the coefficients satisfy $\sum |\lambda_j| < \infty$. If A is a subset of $\partial\Omega$, we define the Hardy space $H^1(A, \sigma_{\epsilon'})$ as the restrictions of A of elements of the Hardy space on $\partial\Omega$.

The behavior of the Green's function and a related Hardy space estimate

We can define a Green's function for the mixed problem and it is Hölder continuous as follows:

$$|M(x,y_1)-M(x,y_2)| \le C\left(\frac{|y_1-y_2|}{|x-y_1|}\right)^{\delta}, \qquad 2|y_1-y_2| < |x-y_1|.$$

This estimate can be used to prove that if u is a solution of the mixed problem with Neumann data an atom and zero Dirichlet data then $|\nabla u|$ decays like $|x|^{-1-\delta}$ at infinity (at least in an average sense). This is the key estimate that we need to show that

$$(\nabla u)^* \in L^1(\sigma_{\epsilon'}).$$

We show that we can solve the mixed problem for data $f_N \in H^1(N, \sigma_{\epsilon'})$ and $df_D/ds \in H^1(D, \sigma_{\epsilon'})$ for $-\epsilon_0 < \epsilon' \le 0$.

Interpolation, an L^p-result

Stromberg and Torchinsky (1989) have established a theorem on interpolation for Hardy spaces which allows change of measure. Thus interpolation between $L^2(\sigma_\epsilon)$ with $\epsilon > 0$ and $L^2(\sigma_{\epsilon'})$ with $\epsilon' < 0$, we conclude that we may solve the mixed problem with data in (unweighted) $L^p(\sigma)$ for $1 < p < p_0$.

Two questions.

1. Is the restriction to $M < 1$ essential?

2. Can we establish existence for L^p, with p near 1 in Lipschitz domains in higher dimensions?

Shen has a method for obtaining weighted estimates that does not rely on complex analysis.