Continuity of a scattering transform for a first-order system.

Russell Brown
Department of Mathematics
University of Kentucky
Lexington, Kentucky
http://www.ms.uky.edu/~rbrown

Special session in inverse problems at the 999th regional meeting of the American Mathematics Society.

Vanderbilt University
Nashville, Tennessee.

October 2004.

A first order system

We consider the system

$$\begin{pmatrix} \partial_{\bar{x}} & 0 \\ 0 & \partial_x \end{pmatrix} \psi - \begin{pmatrix} 0 & q^1 \\ q^2 & 0 \end{pmatrix} \psi = 0, \qquad (1)$$

or more compactly, $D\psi - Q\psi = 0$. Here, $x = x^1 + ix^2$ is a complex variable and ∂_x and $\partial_{\bar{x}}$ are the standard complex derivatives. The potentials q^j are functions on the plane.

Two motivations:

- Connection to inverse conductivity problem.

- Relation to Davey-Stewartson II system

Inverse conductivity problem.

Suppose that u is a solution of div$\gamma\nabla u = 0$ in the plane. We may construct ψ a solution of the system (1) by setting

$$\psi = \sqrt{\gamma}\begin{pmatrix} \partial_x u \\ \partial_{\bar{x}} u \end{pmatrix}, \quad q^1 = \partial_x \log\sqrt{\gamma} \quad \text{and } q^2 = \partial_{\bar{x}} \log\sqrt{\gamma}.$$

Brown and Uhlmann (1997) used this system to establish uniqueness in the inverse conductivity problem in the class of conductivities, γ, which have one derivative in L^p, $p > 2$.

Nachman (1994) had treated conductivities with two derivatives in L^p, $p > 1$. Astala and Päivärinta (2003) have shown that uniqueness holds for conductivities in two-dimensions which are only bounded and measurable. They relate the conductivity equation to the Beltrami equation. The problem in three dimensions was treated by Sylvester and Uhlmann.

Brown and Uhlmann's argument relied on scattering theory that had been developed earlier by Fokas and Ablowitz (1984), Beals and Coifman (1985, 1988) and L.Y. Sung (1994).

This work had shown that a scattering transform (to be defined below) for system (1), could be used to transform the Davey-Stewartson II system to a linear evolution.

Davey-Stewartson.

The Davey-Stewartson II system is a nonlinear system of partial differential equations in two space dimensions which relates a complex valued function $q(x,t)$ and a real function $r(x,t)$.

$$\begin{cases} q_t = iq_{x^1x^2} - 4irq \\ r_{x^1x^1} + r_{x^2x^2} = (|q|^2)_{x^1x^2} \end{cases} \tag{2}$$

These equations were introduced in 1974 to model "the evolution of weakly non-linear waves that travel predominantly in one direction."

Exponentially growing solutions.

We begin with the matrix valued solution to the system

$$\begin{pmatrix} \partial_{\bar{x}} & 0 \\ 0 & \partial_x \end{pmatrix} \psi_0 = 0$$

given by

$$\psi_0(x,z) = \begin{pmatrix} \exp(ixz) & 0 \\ 0 & \exp(-i\bar{x}z) \end{pmatrix}$$

We look for solutions of the system (1), where the potential is not zero, which are of the form

$$\psi = m\psi_0$$

and m approaches the 2×2 identity matrix at infinity.

The $\bar{\partial}$-equation.

Suppose that q^1 and q^2 are nice. If, in addition, the functions q^1 and q^2 are small or satisfy one of the symmetry conditions $q^1 = \pm\bar{q}^2$, then we can construct m and show that m is a solution of the *$\bar{\partial}$-equation* in the variable z:

$$\partial_{\bar{z}} m(x,z) = m(x,\bar{z})S(z)A(x,-\bar{z}). \tag{3}$$

where

$$A(x,z) = \begin{pmatrix} \exp(ix\bar{z} + i\bar{x}z) & 0 \\ 0 & \exp(-i\bar{x}\bar{z} - ixz) \end{pmatrix} = \begin{pmatrix} a^1(x,z) & 0 \\ 0 & a^2(x,z) \end{pmatrix}$$

is a diagonal matrix with entries which are bounded exponentials.

The scattering transform.

The function $S(z)$ which appears in the $\bar{\partial}$-equation is an off-diagonal matrix valued function which we call the *scattering transform* of the potential Q.

One can also show that S appears in the asymptotic expansion of the solutions m.

Scattering transform II.

For our purposes, the most useful expression for S is s

$$\frac{1}{2\pi}\int_{\mathbf{C}}\begin{pmatrix} 0 & -ia^1(x,-z)q^1(x)m^{22}(x,z) \\ ia^2(x,-z)q^2(x)m^{11}(x,z) & 0 \end{pmatrix} dx.$$

Beals and Coifman show that when one of the conditions $Q = \pm Q^*$ holds, we have the remarkable identity

$$\int \text{trace } QQ^* dx = \int \text{trace } SS^* dz. \tag{4}$$

If we believe that the main term of m^{jj} is 1, we can see that scattering map $Q \to S$ is a kind of non-linear Fourier transform and the identity (4) is a Plancherel identity for this transform.

Some questions.

- Is the map $Q \to S$ defined for all S in L^2?

- Is this map continuous on L^2?

Sung (1994) has studied that scattering transform for potentials in $L^1 \cap L^\infty$.

Constructing solutions m.

To proceed, we indicate how to construct S when Q is not too large, but without the symmetry assumptions $Q = \pm Q^*$.

Recall that $m\psi_0$ is a matrix valued solution of the system $(D - Q)(m\psi_0) = 0$. Commuting the exponentials, we can show that $(D_z - Q)m = 0$ where $D_z = E_z^{-1} D E_z$ and the map E_z acts on matrix valued functions by

$$E_z f = f^{\text{diag}} + A(\cdot, -z) f^{\text{off}}.$$

Constructing solutions m.

The solutions m can be written

$$m = \sum_{j=0}^{\infty} (D_z^{-1} Q)^j (1)$$

where 1 denotes the 2×2 identity matrix. The inverse of D_z is easy to write down since the matrix A is invertible and we may use the Cauchy transform to invert D.

The scattering data, again.

If we substitute the series for m into the integral expression for the scattering data, we may represent S by

$$S = \sum_{j=0}^{\infty} S_j.$$

where, for example,

$$S_k^{12}(z)$$
$$= \frac{1}{\pi^{2k}} \int_{\mathbf{C}^{2k+1}} a^1(-x_0 + x_1 - x_2 + x_3 \ldots + x_{2k-1} - x_{2k}, z)$$
$$\times \frac{Q^{12}(x_0)Q^{21}(x_1)\ldots Q^{21}(x_{2k-1})Q^{12}(x_{2k})}{(\bar{x}_0 - \bar{x}_1)(x_1 - x_2)\ldots(\bar{x}_{2k-2} - \bar{x}_{2k-1})(x_{2k-1} - x_{2k})}\, dx_0 \ldots dx_{2k}.$$

Duality.

To estimate these expressions, we use duality. Let T be in L^2, then

$$\int_{\mathbf{C}} T(z) S_k^{12}(z)\, dz$$
$$= \frac{1}{\pi^{2k}} \int_{\mathbf{C}^{2k+2}} T(z) a^1(-x_0 + x_1 - x_2 + x_3 \ldots + x_{2k-1} - x_{2k}, z)$$
$$\times \frac{Q^{12}(x_0) Q^{21}(x_1) \ldots Q^{21}(x_{2k-1}) Q^{12}(x_{2k})}{(\bar{x}_0 - \bar{x}_1)(x_1 - x_2) \ldots (\bar{x}_{2k-2} - \bar{x}_{2k-1})(x_{2k-1} - x_{2k})}\, dz\, dx_0 \ldots dx_{2k}.$$

If we integrate in z this gives the Fourier transform of T:

$$\frac{1}{\pi^{2k}} \int_{\mathbf{C}^{2k+1}} \widehat{T}(2(x_0 - x_1 + x_2 - \ldots - x_{2k-1} + x_{2k})) \\ \times \frac{Q^{12}(x_0) Q^{21}(x_1) \ldots Q^{21}(x_{2k-1}) Q^{12}(x_{2k})}{(\bar{x}_0 - \bar{x}_1)(x_1 - x_2) \ldots (\bar{x}_{2k-2} - \bar{x}_{2k-1})(x_{2k-1} - x_{2k})} \, dx_0 \ldots dx_{2k}.$$

Thus we are interested in the following multi-linear expressions.

$$I_k(t, q_0, \ldots, q_{2k}) \\ = \int_{\mathbf{C}^{2k+1}} \frac{t(x_0 - x_1 + x_2 - \ldots - x_{2k-1} + x_{2k}) \prod_{j=0}^{2k} q_j(x_j)}{|x_0 - x_1||x_1 - x_2| \cdots |x_{2k-1} - x_{2k}|} \, dx_0 \ldots dx_{2k}.$$

We can estimate these expressions using :

- Hölder's inequality

- L^p-mapping properties for first-order Riesz potentials,

$$R(f)(x) = \frac{1}{\pi} \int \frac{f(y)}{|x-y|} \, dy$$

  including Lieb's (1984) sharp result $R : L^{4/3} \to L^4$ with norm 1.

- Some persistence.

Main Lemma

Lemma 1 *If* $s = 4$ *or* $s = 4/3$*, then for each* $\epsilon > 0$*, there is constant* C_ϵ *so that*

$$|I_k(t, q_0, \ldots, q_{2k})| \leq C_\epsilon \pi^{2k} (1+\epsilon)^{2k} \|t\|_2 \|q_0\|_s \|q_{2k}\|_{s'} \prod_{j=1}^{2k-1} \|q_j\|_2 \tag{5}$$

From this, interpolation implies that we may replace s by 2.

The main theorem.

Finally, summing the series for S, we can show:

Theorem 1 *If Q is an off-diagonal matrix with $\|Q\|_2 \leq \sqrt{2}$, then corresponding scattering data satisfies*

$$\|S\|_2 \leq C\|Q\|_2.$$

If Q $\tilde{Q}$ are two such potentials with corresponding data S and $\tilde{S}$, then

$$\|S - \tilde{S}\|_2 \leq C\|Q - \tilde{Q}\|_2.$$

The continuity is obtained by writing the difference of S and $\tilde{S}$ as a sum of multilinear expressions.

Some questions:

- If Q is in L^2 and satisfies one of $Q = \pm Q^*$, can we establish continuity?

- In what sense does the resulting function Q satisfy the Davey-Stewartson II system? Note that it is not so clear how to define

$$r = \Delta^{-1}(|q|^2)_{x^1 x^2}$$

when q is only in L^2.